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Applications of the Operator ${}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,f\theta\right)$

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Abstract

In this paper, we construct the q-exponential operator ${}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,f\theta\right)$. We use the operator ${}_3\phi_2$ to obtain an extension of Euler identities, Ramanujan's sum, q-Chu- Vanermonde summation formula and we give some other identities. Also we use the operator ${}_3\phi_2$ to get an extension of the Ramanujan's identity, the Askey beta integral, Ramanujan's beta integral and we give some other integrals formulas.

1- Introduction

In this paper we will use the standard notations for basic hypergeometric series given in [5], we assume that $|q| < 1$.

Definition 1.1. [5]. *Let a be a complex variable. The q-shifted factorial is defined by*

$$(a;q)_n = \begin{cases} 1, & if \;\; n=0, \\ \prod_{k=0}^{n-1}(1-aq^k), & if \;\; n=1,2,\ldots. \end{cases}$$

We define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The following notation is used for the multiple q-shifted factorials:

$$(a_1,a_2,\cdots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \qquad n=0,1,2,\ldots$$

$$a_1,a_2,\cdots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

Definition 1.2 [5]. *The generalized basic hypergeometric series is defined by*

$${}_r\phi_s\left(\begin{matrix} a_1,a_2,\cdots,a_r \\ b_1,b_2,\cdots,b_s \end{matrix};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

where $r,s \in \mathbb{N}$; $a_1,\ldots,a_r \in C$; $b_1,\ldots,b_s \in C\backslash\{q^{-k}, k \in N\}$ are assumed to be such that none of the denominator factors evaluate to zero. This series converges absolutely for all x if $r \le s$ and for $|x| < 1$ if $r = s+1$.

The case $r = s + 1$ is the most important class of series

$$_{s+1}\phi_s\left(\begin{matrix} a_1, a_2, \cdots, a_{s+1} \\ b_1, b_2, \cdots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_{s+1}; q)_n}{(q, b_1, \cdots, b_s; q)_n} x^n, \quad |x| < 1.$$

The general bilateral basic hypergeometric series is given by:

$$_r\psi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} x^n, \quad |x| < 1.$$

Ramanujan's sum

$$_1\psi_1(a; b; q, x) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n = \frac{(q, b/a, ax, q/ax; q)_\infty}{(b, q/a, x, b/ax)_\infty}. \tag{1.1}$$

Definition 1.3 [5]. *For $n \in N$, the q-binomial coefficient is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q; q)_n}{(q; q)_k\, (q; q)_{n-k}}, & if \;\; 0 \le k \le n; \\ 0, & otherwise. \end{cases}$$

In this paper, we will use the following identities ([5]):

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}. \tag{1.2}$$

$$(aq^{-n}; q)_n = (q/a; q)_n (-a/q)^n q^{-\binom{n}{2}}. \tag{1.3}$$

$$(a; q)_n = (q^{1-n}/a; q)_n (-a)^n q^{\binom{n}{2}}. \tag{1.4}$$

One of the most important identities is the Cauchy identity ([5])

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1. \tag{1.5}$$

Euler found the following special case of Cauchy identity ([5]):

$$\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \quad |x| < 1. \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} = (-x; q)_\infty. \tag{1.7}$$

Hein's q-Gauss summation formula is ([5])

$$_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, c/ab\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \tag{1.8}$$

The q-Chu-Vandermonde summation formula ([5]):

$$_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q\right) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \tag{1.9}$$

Definition 1.4 [1, 9]. *The q-differential operator θ is defined by*

$$\theta\{f(x)\} = \frac{f(q^{-1}x) - f(x)}{q^{-1}x}. \tag{1.10}$$

Definition 1.5 [9]. *The Leibniz rule for θ is*

$$\theta^n\{f(x)\,g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(x)\}\theta^{n-k}\{g(xq^{-k})\}. \tag{1.11}$$

The following identities are easy to prove:

Theorem 1.6 [4, 12]. *Let θ be defined as in* (1.10), *then*

$$\theta^k\{x^n\} = (-1)^k x^{n-k} q^k (q^{-n};q)_k.$$

$$\theta^k\{(xt;q)_\infty\} = (-t)^k (xt;q)_\infty.$$

$$\theta^k\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\} = t^k q^{-\binom{k}{2}} (v/t;q)_k \frac{(xv;q)_\infty}{(xtq^{-k};q)_\infty}, \quad |xt| < 1.$$

Definition 1.7 [4]. *The q-exponential operator $E(b\theta)$ is defined by*

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}.$$

Theorem 1.8 *Let θ be defined as in* (1.10), *then*

$$E(b\theta)\{(at;q)_\infty\} = (at,bt;q)_\infty.$$

$$E(b\theta)\{(as,at,;q)_\infty\} = \frac{(as,at,bs,bt;q)_\infty}{(abst/q;q)_\infty}. \tag{1.12}$$

Based on the q-Chu-Vandermonde summation formula (1.9), Zhang and Yang [13] considered the finite q-exponential operator ${}_2\mathcal{T}_1\left(\begin{matrix} q^{-N}, \mathrm{v} \\ w \end{matrix}; q, t\theta\right)$ with two parameters as follows:

Definition 1.9 [13]. *The finite q-exponential operator ${}_2\mathcal{T}_1\left(\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, t\theta\right)$ is defined by*

$${}_2\mathcal{T}_1\left(\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, t\theta\right) = \sum_{n=0}^{N} \frac{(q^{-N},v;q)_n}{(q,w;q)_n} (t\theta)^n. \tag{1.13}$$

Zhang and Yang [13] proved the following result:

Theorem 1.10 [13]. *Let ${}_2\mathcal{T}_1\left(\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, t\theta\right)$ be defined as in* (1.13), *then we have*

$${}_2\mathcal{T}_1\left(\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, t\theta\right)\{(xb;q)_\infty\} = (xb;q)_\infty \; {}_2\psi_1\left(\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, -tb\right).$$

Inspired by the basic hypergeometric series ${}_2\phi_1$, Li and Tan [8] introduced the generalized q-exponential operator $\mathbb{E}\left[\begin{matrix} \boldsymbol{u}, \boldsymbol{v} \\ w \end{matrix} | q; t\theta\right]$ with three parameters as follows:

Definition 1.11 [8]. *The generalized q-exponential operator $\mathbb{E}\left[\begin{matrix} \boldsymbol{u}, \boldsymbol{v} \\ w \end{matrix} | q; t\theta\right]$ is defined by*

$$\mathbb{E}\left[\begin{matrix} \boldsymbol{u}, \boldsymbol{v} \\ w \end{matrix} | q; t\theta\right] = \sum_{n=0}^{\infty} \frac{(u,v;q)_n}{(q,w;q)_n} (t\theta)^n. \tag{1.14}$$

Li and Tan [8] proved the following result:

Theorem 1.12 [8]. *Let $\mathbb{E}\left[\begin{matrix} \boldsymbol{u}, \boldsymbol{v} \\ w \end{matrix} | q; t\theta\right]$ be defined as in* (1.14), *then we have*

$$\mathbb{E}\left[\begin{matrix}\boldsymbol{u},\boldsymbol{v}\\ w\end{matrix}|q;\frac{w}{uv\theta}\right]\{x^n\} = x^n\ {}_3\phi_2\left(\begin{matrix}q^{-N},u,v\\ w,0\end{matrix};q,-\frac{qw}{uvx}\right).$$

$$\mathbb{E}\left[\begin{matrix}\boldsymbol{u},\boldsymbol{v}\\ w\end{matrix}|q;t\theta\right]\left\{\frac{(xa;q)_\infty}{(xb;q)_\infty}\right\} = \frac{(xa;q)_\infty}{(xb;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}u,v,a/b\\ w,q/xb\end{matrix};q,;q,-qt/x\right), \tag{1.15}$$

where θ acts on x.

Thomae [10, 11] Jackson [6, 7] introduced the q-integral

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n$$

and Jackson gave the more general definition

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt,$$

where

$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

Jackson also defined an integral on $(0,\infty)$ by

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n.$$

The bilateral q-integral is defined by

$$\int_{-\infty}^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} [f(q^n)+f(-q^n)]q^n.$$

Askey beta integral is given by ([2]):

$$\int_{-\infty}^{\infty}\frac{(xt,yt;q)_\infty}{(-wt,ut;q)_\infty}d_qt = \frac{2(1-q)(q^2;q^2)_\infty^2(wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty(w^2,u^2,q^2/w^2;q^2)_\infty(-xy/uwq;q)_\infty}. \tag{1.16}$$

Ramanujan's identity (i) is given by ([3])

$$\int_{-\infty}^{\infty}\frac{e^{-x^2+2mx}}{(sq^{1/2}e^{2ikx},tq^{1/2}e^{-2ikx};q)_\infty}dx = \sqrt{\pi}e^{m^2}\frac{(-sqe^{2mki},-tqe^{-2mki};q)_\infty}{(stq;q)_\infty}. \tag{1.17}$$

The Ramanujan's beta integral is given by

$$\int_0^\infty t^{x-1}\frac{(-yt;q)_\infty}{(t;q)_\infty}dt = \frac{\pi}{\sin(\pi x)}\frac{(q^{1-x},y;q)_\infty}{(q,yq^{-x};q)_\infty}. \tag{1.18}$$

2. The q-Exponential Operator and its Operator Identities

In this section, based on the basic hypergeometric series ${}_3\phi_2$, we define a q-exponential operator with five parameters ${}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)$ and obtain some its operator identities.

Definition 2.1. *The q-exponential operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)$ is defined as follows:*

$${}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right) = \sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_n}(f\theta)^k. \tag{2.1}$$

Note that the finite q-exponential operator defined by Zhang and Yang [13] can be considered as special case of our operator for $a = q^{-N}$, $b = v$, $d = w$ and $c = e = 0$. Also, the generalized q-exponential operator defined by Li and Tan in [8] can be considered as special case of our operator for $a = u$, $b = v$, $c = 0$, $d = w$, $e = 0$ and $f = t$.

Theorem 2.2 *We have*

$$ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\left\{\frac{(xu;q)_\infty}{(xs;q)_\infty}\right\} = \frac{(xu;q)_\infty}{(xs;q)_\infty}\ {}_4\phi_3\left(\begin{matrix}a,b,c,u/s\\ d,e,q/xs\end{matrix};q,-qf/x\right). \tag{2.2}$$

provided that $\max\{|f/x|,|xs|\} < 1$.

Proof.

$$\begin{aligned} &{}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\left\{\frac{(xu;q)_\infty}{(xs;q)_\infty}\right\}\\ &= \sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\theta^k\left\{\frac{(xu;q)_\infty}{(xs;q)_\infty}\right\}\\ &= \sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k q^{-k(k-1)}s^k\frac{(u/s;q)_k(xu;q)_\infty}{(xsq^{-k};q)_\infty}\\ &= \frac{(xu;q)_\infty}{(xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k q^{-k(k-1)}s^k\frac{(u/s;q)_k}{(-xs/q)^k q^{-k(k-1)}(q/xs;q)_k}\\ &= \frac{(xu;q)_\infty}{(xs;q)_\infty}\ {}_4\phi_3\left(\begin{matrix}a,b,c,u/s\\ d,e,q/xs\end{matrix};q,-qf/x\right). \end{aligned}$$

■

Note that if $c = e = 0$ in (2.2) we get equation (1.15) proved by Li and Tan [8].

Setting $u = 0$ in (2.2), we get the following corollary:

Corollary 2.2.1. *We have*

$$ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\left\{\frac{1}{(xs;q)_\infty}\right\} = \frac{1}{(xs;q)_\infty}\ {}_4\phi_3\left(\begin{matrix}a,b,c,0\\ d,e,q/xs\end{matrix};q,-qf/x\right), \tag{2.3}$$

provided that $\max\{|xs|,|qf/x|\} < 1$.

Setting $s = 0$ in (2.2), we get the following corollary:

Corollary 2.2.2. *We have*

$$ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\{(xu;q)_\infty\} = (xu;q)_\infty\ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,-fu\right), \tag{2.4}$$

provided that $|fu| < 1$.

Theorem 2.3 *We have*

$$\begin{aligned} &{}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\{(xs,xt,q)_\infty\}\\ &= (xs,xt,q)_\infty\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b,c;q)_{k+j}}{(d,e;q)_{k+j}}\frac{(-ftq^{-j})^k}{(q;q)_k}\frac{(q/xt;q)_j(ftsx/q)^j q^{-\binom{j}{2}}}{(q;q)_j}. \end{aligned} \tag{2.5}$$

Proof. By using Leibniz rule (1.11), we have

$$ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\{(xs,xt;q)_\infty\}$$

$$=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\theta_q^j\{(xs;q)_\infty\}\theta^{k-j}\{(xtq^{-j};q)_\infty\}$$

$$=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(-s)^j(xs;q)_\infty(-tq^{-j})^{k-j}(xtq^{-j};q)_\infty$$

$$=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(-s)^j(xs;q)_\infty(-tq^{-j})^{k-j}(xtq^{-j};q)_j(xt;q)_\infty.$$

By using (1.4) and (1.2) we get

$$=(xs,xt;q)_\infty\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b,c;q)_{k+j}f^{k+j}}{(q,d,e;q)_{k+j}}\frac{(-tq^{-j})^k}{(q;q)_k}\frac{(q/xt;q)_j(-xt)^jq^{-j}q^{-\binom{j}{2}}}{(q;q)_j}$$

$$=(xs,xt;q)_\infty\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b,c;q)_{k+j}}{(d,e;q)_{k+j}}\frac{(-ftq^{-j})^k}{(q;q)_k}\frac{(q/xt;q)_j(ftsx/q)^jq^{-\binom{j}{2}}}{(q;q)_j}.\qquad\blacksquare$$

Note that, setting $a=0$, $b=0$, $c=0$, $d=0$, $e=0$ and $f=cq^{(k+j-1)/2}$ in (2.5) and by using Euler identity (1.6) and Cauchy identity (1.5), we get the Theorem 2.11. obtained by Chen and Liu [4].

Theorem 2.4. *Let the operator* ${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)$ *be defined as in* (2.1) *and n is a nonnegative integer, then*

$$ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\{(xt,q)_\infty x^n\}$$
$$=(xt,q)_\infty x^n\sum_{j=0}^{\infty}\sum_{k=0}^{n}\frac{(a,b,c;q)_{k+j}(q^{-n};q)_k(-qf/x)^k(-ftq^{-n})^j}{(d,e;q)_{k+j}(q;q)_k(q;q)_j}.\qquad(2.6)$$

Proof. From definition of the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)$, we have

$${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\{(xt,q)_\infty x^n\}=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\theta^k\{x^n(xt,q)_\infty\}.$$

By using Leibniz rule (1.11), we have

$$\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\theta^k\{(xt,q)_\infty x^n\}$$

$$=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\theta^j\{(xt;q)_\infty\}\theta^{k-j}\{(xq^{-j})^n\}$$

$$=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(-t)^j(xt;q)_\infty q^{-nj}(-1)^{(k-j)}q^{(k-j)}(q^{-n};q)_{k-j}x^{n-(k-j)}$$

$$=x^n(xt,q)_\infty\sum_{j=0}^{\infty}\sum_{k=0}^{n}\frac{(a,b,c;q)_{k+j}(q^{-n};q)_k(-qf/x)^k(-ftq^{-n})^j}{(d,e;q)_{k+j}(q;q)_k(q;q)_j}.\qquad\blacksquare$$

Note that, setting $a=b=c=d=0$ and $f=d$ in (2.6) and then using equations (1.7), (1.2) and (1.4), we get Corollary 2.4. obtained by Zhang and Liu [14].

Setting $t=0$ in (2.6), we get the following corollary:

Corollary 2.4.1. *We have*

$$ {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; q, f\theta\right)\{x^n\} = x^n \ {}_4\phi_3\left(\begin{matrix} a,b,c,q^{-n} \\ d,e,0 \end{matrix}; q, -qf/x\right). \quad (2.7) $$

When $c = e = 0$ and $f = d/ab$ in (2.7) we obtain equation (1.15) obtained by Li and Tan [8].

3. Applications in q-Identities

In this section, we use the operator ${}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; q, f\theta\right)$ to give an extension to some wellknown q-identities such as: Euler identities (1.6), (1.7), Ramanujan's sum (1.1), q-Chu-Vanermonde summation formula (1.9) and we give some other identities.

3.1 Extension of Euler Identities

Theorem 3.1. (Extension of Euler identity (1.7)). *We have*

$$ \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} \ {}_4\phi_3\left(\begin{matrix} a,b,c,q^{-n} \\ d,e,0 \end{matrix}; q, -qf/x\right) = (-x;q)_\infty \ {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; q, f\right). \quad (3.1) $$

Proof. Recalling Euler identity, we have

$$ \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x;q)_\infty. $$

Applying the operator ${}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; q, f\theta\right)$ to both sides of the above equation with respect to the parameter x, we get

$$ \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} \ {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; q, f\theta\right)\{x^n\} = {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; q, f\theta\right)\{(-x;q)_\infty\}. $$

By using (2.7) and (2.4), we get

$$ \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} \ {}_4\phi_3\left(\begin{matrix} a,b,c,q^{-n} \\ d,e,0 \end{matrix}; q, -qf/x\right) = (-x;q)_\infty \ {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; q, f\right). \quad \blacksquare $$

Theorem 3.2. (Cauchy identity). *We have*

$$ \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}. $$

Proof. Setting $b = c = e = 0$, $f = -x$, in equation (3.1), we get

$$ \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} \ {}_2\phi_1\left(\begin{matrix} q^{-n},a \\ d \end{matrix}; q, q\right) = (-x;q)_\infty \ {}_2\phi_1\left(\begin{matrix} a,0 \\ d \end{matrix}; q, -x\right). $$

By using (1.9), we get

$$ \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} a^n \frac{(d/a;q)_n}{(d;q)_n} = (-x;q)_\infty \ {}_2\phi_1\left(\begin{matrix} a,0 \\ d \end{matrix}; q, -x\right) $$

$$ \sum_{n=0}^{\infty} \frac{(d/a;q)_n q^{\binom{n}{2}} (xa)^n}{(q,d;q)_n} = (-x;q)_\infty \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,d;q)_n} (-x)^n. $$

Replacing (d,x) by $(0,-x)$ in above equation to have:

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(-xa)^n}{(q;q)_n}=(x;q)_\infty\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\;x^n.$$

by using Euler identity we obtain

$$\frac{(xa;q)_\infty}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\;x^n. \qquad \blacksquare$$

Setting $c=e=0, f=d/ab$ in (3.1) and then using the Hein's q-Gauss summation formula (1.8), we get the following corollary:

Corollary 3.2.1. *We have*

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{(q;q)_n}\;{}_3\phi_2\left(\begin{matrix}q^{-n},a,b\\ d,0\end{matrix};q,-dq/abx\right)=(-x;q)_\infty\frac{(d/a,d/b;q)_\infty}{(d,d/ab;q)_\infty}.$$

Theorem 3.3. (Extension of Euler identity (1.6)). *We have*

$$\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}\;{}_4\phi_3\left(\begin{matrix}a,b,c,q^{-n}\\ d,e,0\end{matrix};q,-qf/x\right)=\frac{1}{(x;q)_\infty}\;{}_4\phi_3\left(\begin{matrix}a,b,c,0\\ d,e,\frac{q}{x}\end{matrix};q,-\frac{fq}{x}\right). \tag{3.2}$$

Proof. From Euler identity (1.6), we have

$$\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}=\frac{1}{(x;q)_\infty}.$$

Applying the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)$ with respect to x , we have

$$\sum_{n=0}^{\infty}\frac{1}{(q;q)_n}\;{}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\{x^n\}=\;{}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\left\{\frac{1}{(x;q)_\infty}\right\}.$$

By using (2.7) and (2.3) the proof is complete . $\blacksquare$

3.2. Extension of Ramanujan's Sum

Theorem 3.4. (Extension of Ramanujan's sum (1.1)). *We have*

$$\sum_{k=0}^{n}\frac{(s;q)_n}{(t;q)_n}\;x^n\;{}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,-fq^n\right)$$
$$=\frac{(q,sx,q/sx,t/s;q)_\infty}{(t,q/s,x,t/sx;q)_\infty}\;{}_4\phi_3\left(\begin{matrix}a,b,c,x\\ d,e,qsx/t\end{matrix};q,-qf/t\right). \tag{3.3}$$

Proof. From Ramanujan's sum (1.1) we have

$$\sum_{n=0}^{\infty}\frac{(s;q)_n}{(t;q)_n}x^n=\frac{(q,t/s,sx,q/sx;q)_\infty}{(t,q/s,x,t/sx;q)_\infty}$$

$$\sum_{n=0}^{\infty}(s;q)_n x^n\frac{(tq^n;q)_\infty}{(t;q)_\infty}=\frac{(q,t/s,sx,q/sx;q)_\infty}{(t,q/s,x,t/sx;q)_\infty}$$

$$\sum_{n=0}^{\infty}(s;q)_n x^n(tq^n;q)_\infty=\frac{(q,t/s,sx,q/sx;q)_\infty}{(q/s,x,t/sx;q)_\infty}.$$

Applying the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)$ to both sides of the above equation with respect to the parameter t, we get

$$\sum_{n=0}^{\infty}(s;q)_n x^n\ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\{(tq^n;q)_\infty\}$$
$$=\frac{(q,sx,q/sx;q)_\infty}{(q/s,x;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\left\{\frac{(t/s,q)_\infty}{(t/sx;q)_\infty}\right\}.$$

Now by using the relation (2.4) and (2.2) we get (3.3). ■

Corollary 3.4.1. *We have*

$$\sum_{n=0}^{\infty}\frac{(s;q)_n}{(t;q)_n}(tq^{n-m};q)_m x^n=\frac{(q,sx,q/sx,t/s;q)_\infty}{(t,q/s,x,t/sx;q)_\infty}\frac{(qs/t;q)_m}{(qsx/t;q)_m}x^m.$$

Proof. letting $a=q^{-m}$,$b=c=d=e=0$ and $f=-t$ in (3.3), we have

$$\sum_{n=0}^{\infty}\frac{(s;q)_n}{(t;q)_n}x^n\sum_{k=0}^{m}\frac{(q^{-m};q)_k}{(q;q)_k}(tq^n)^k=\frac{(q,sx,q/sx,t/s;q)_\infty}{(t,q/s,x,qsx/t;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}q^{-m},x\\qsx/t\end{matrix};q,q\right)$$
$$\sum_{n=0}^{\infty}\frac{(s;q)_n}{(t;q)_n}x^n\frac{(tq^{n-m};q)_\infty}{(tq^n;q)_\infty}=\frac{(q,sx,q/sx,t/s;q)_\infty}{(t,q/s,x,t/sx;q)_\infty}\frac{(qs/t;q)_m}{(qsx/t;q)_m}x^m$$
$$\sum_{n=0}^{\infty}\frac{(s;q)_n}{(t;q)_n}(tq^{n-m};q)_m x^n=\frac{(q,sx,q/sx,t/s;q)_\infty}{(t,q/s,x,t/sx;q)_\infty}\frac{(qs/t;q)_m}{(qsx/t;q)_m}x^m.$$ ■

Setting $b=c=e=0, f=-x$, in (3.2) and using the q-Chu-Vandermonde summation formula (1.9), we get the following corollary:

Corollary 3.4.2. *We have*

$${}_2\phi_1\left(\begin{matrix}d/a,0\\d\end{matrix};q,xa\right)=\frac{1}{(x;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}a,0,0\\d,q/x\end{matrix};q,q\right).$$

3.3 Extension of the q-Chu-Vandermonde Summation Formula

Theorem 3.5. (Extension of the q-Chu-Vandermonde summation formula (1.9)). *We have*

$$\sum_{j=0}^{n}\frac{(q^{-n},x;q)_j}{(q,y;q)_j}q^j\ {}_4\phi_3\left(\begin{matrix}a,b,c,q^jx\\d,e,qx/y\end{matrix};q,-qf/y\right)$$
$$=x^n\frac{(y/x;q)_n}{(y;q)_n}\ {}_4\phi_3\left(\begin{matrix}a,b,c,xq^{2n}\\d,e,q^{1+n}x/y\end{matrix};q,-qf/y\right).\qquad(3.4)$$

Proof. Recalling the q -Chu-Vandermonde summation formula

$$\sum_{j=0}^{n}\frac{(q^{-n},x;q)_j}{(q,y;q)_j}q^j=x^n\frac{(y/x;q)_n}{(y;q)_n}$$
$$\sum_{j=0}^{n}\frac{(q^{-n},x;q)_j}{(q;q)_j}q^j\frac{(yq^j;q)_\infty}{(y/x;q)_\infty}=x^n\frac{(yq^n;q)_\infty}{(y/xq^n;q)_\infty}.$$

Applying the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)$ with respect to y, we obtain

$$\sum_{j=0}^{n}\frac{(q^{-n},x;q)_j}{(q;q)_j}q^j\ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\left\{\frac{(yq^j;q)_\infty}{(y/x;q)_\infty}\right\}=x^n\ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\left\{\frac{(yq^n;q)_\infty}{(y/xq^n;q)_\infty}\right\}.$$

Then by using (2.2) the proof is complete . ■

4. Applications q-Integrals

In this section, we use the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)$ to obtain an extension of the Askey beta integral (1.16), the Ramanujan's identity (1.17), , Ramanujans beta integral (1.18) and we give some other integrals formulas.

4.1. Extension of the Askey Beta Integral

Theorem 4.1. (Extension of the Askey Beta Integral (1.16)). *We have*

$$\int_{-\infty}^{\infty}\frac{(xt,yt;q)_\infty}{(-wt,ut;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b,c;q)_{k+j}}{(d,e;q)_{k+j}}\frac{(fyq^{-1-j}/wu)^k}{(q;q)_k}\frac{(-uwq^2/xy;q)_j}{(q;q)_j}\left(\frac{-fyxt}{wuq^2}\right)^j q^{-\binom{j}{2}}d_qt$$
$$=\frac{2(1-q)(q^2;q^2)_\infty^2(wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty(w^2,u^2,q^2/w^2;q^2)_\infty}$$
$$\times\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b,c;q)_{k+j}}{(d,e;q)_{k+j}}\frac{(fq^{-j}/w)^k}{(q;q)_k}\frac{(-qw/x;q)_j}{(q;q)_j}\left(\frac{-fx}{uwq}\right)^j q^{-\binom{j}{2}}. \qquad (4.1)$$

Proof. From Askey beta integral (1.16) we have:

$$\int_{-\infty}^{\infty}\frac{(xt,yt;q)_\infty}{(-wt,ut;q)_\infty}d_qt=\frac{2(1-q)(q^2;q^2)_\infty^2(wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty(w^2,u^2,q^2/w^2;q^2)_\infty(-xy/uwq;q)_\infty}.$$

multiplying both sides of the above equation by $(-xy/uwq;q)_\infty$ and then applying the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)$ with respect to x, we get

$$\int_{-\infty}^{\infty}\frac{(yt;q)_\infty}{(-wt,ut;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\{(xt,-xy/uwq;q)_\infty\}d_qt$$
$$=\frac{2(1-q)(q^2;q^2)^2{}_\infty(wu,q/wu,y/u,-y/u;q)_\infty}{(q;q)_\infty(w^2,u^2,q^2/w^2;q^2)_\infty}\ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,f\theta\right)\{(x/u,-x/w;q)_\infty\}.$$

Using equation (2.5) on both sides of above equation, we get the required result. ■

Note that setting $a=b=c=d=e=0$ and $f=cq^{(k+j-1)/2}$ in (4.1) and then using Cauchy identity and Euler identity we get Theorem 6.3 obtained by Chen and Liu [4].

Corollary 4.4.1. *We have*

$$\int_{-\infty}^{\infty}\frac{(xt,yt;q)_\infty}{(-wt,ut;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b;q)_{k+j}}{(u;q)_{k+j}}\frac{(yq^{-1-j}/ab)^k}{(q;q)_k}\frac{(-uwq^2/xy;q)_j}{(q;q)_j}\left(\frac{-yxt}{abq^2}\right)^j q^{-\binom{j}{2}}d_qt$$

$$= \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty} \frac{(u/a,u/b,q)_\infty}{(u,u/ab;q)_\infty}$$

$$\times\ {}_3\phi_2\left(\begin{matrix} a,b,-\frac{qw}{x} \\ \frac{dqab}{u},0 \end{matrix};q,\frac{x}{u}\right). \tag{4.2}$$

Proof. Setting $c = e = 0$, $d = u$ and $f = uw/ab$ in (4.1) we have

$$L.H.S. = \int_{-\infty}^{\infty} \frac{(xt,yt;q)_\infty}{(-wt,ut;q)_\infty} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a,b;q)_{k+j}}{(u;q)_{k+j}} \frac{(yq^{-1-j}/ab)^k}{(q;q)_k} \frac{(-uwq^2/xy;q)_j}{(q;q)_j} (\frac{-yxt}{abq^2})^j q^{-\binom{j}{2}} d_q t.$$

$$R.H.S. = \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b,-qw/x;q;q)_j}{(q,u;q)_j} (\frac{-x}{abq})^j q^{-j_2} \sum_{k=0}^{\infty} \frac{(aq^j,bq^j;q)_k (uq^{-j}/ab)^k}{(q,q^j u;q)_k}$$

$$= \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b,-qw/x;q;q)_j}{(q,u;q)_j} (\frac{-x}{abq})^j q^{-j_2}\ {}_2\phi_1\left(\begin{matrix} aq^j,bq^j \\ uq^j \end{matrix};q,uq^{-j}/ab\right)$$

$$= \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b,-qw/x;q;q)_j}{(q,u;q)_j} (\frac{-x}{abq})^j q^{-j_2} \frac{(u/a,u/b;q)_\infty}{(uq^j,uq^{-j}/ab;q)_\infty} \quad \text{(by using (1.8))}$$

$$= \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty} \frac{(u/a,u/b,q)_\infty}{(u,u/ab;q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b,-qw/x;q)_j}{(q;q)_j} (\frac{-x}{abq})^j q^{-j_2} \frac{1}{(uq^{-j}/ab;q)_j} \quad \text{(by using (1.2))}$$

$$= \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty} \frac{(u/a,u/b,q)_\infty}{(u,u/ab;q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b,-qw/x;q)_j}{(q;q)_j} (\frac{-x}{abq})^j q^{-j_2} \frac{1}{(-1)^j q^{-j_2} \left(\frac{u}{abq}\right)^j (qab/u;q)_j}$$

$$= \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty} \frac{(u/a,u/b,q)_\infty}{(u,u/ab;q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b,-qw/x;q)_j}{(q,qab/u;q)_j} (\frac{x}{u})^j$$

$$= \frac{2(1-q)(q^2;q^2)_\infty^2 (wu,q/wu,x/u,-x/w,y/u,-y/u;q)_\infty}{(q;q)_\infty (w^2,u^2,q^2/w^2;q^2)_\infty} \frac{(u/a,u/b,q)_\infty}{(u,u/ab;q)_\infty}$$

$$\times\ {}_3\phi_2\left(\begin{matrix} a,b,-qw/x \\ qab/u,0 \end{matrix};q,x/u\right). \qquad \blacksquare$$

4.2. Extension of Ramanujan's Identity

Theorem 4.2. (Extension of the Ramanujan's Identity (1.17)). *We have*

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(sqe^{1/2}e^{2ikx},tqe^{1/2}e^{-2ikx};q)_\infty}\ {}_4\phi_3\left(\begin{matrix} a,b,c,0 \\ d,e,\frac{q^{1/2}}{s}e^{-2kix} \end{matrix};q,-qf\right) dx$$

$$= \frac{\sqrt{\pi}e^{m^2}(-sqe^{2mki},-tqe^{-2mki};q)_\infty}{(stq;q)_\infty}\ {}_4\phi_3\left(\begin{matrix}a,b,c,\frac{-e^{2mki}}{t}\\ d,e,1/ts\end{matrix};q,-qf/s\right), \tag{4.3}$$

provided that $\max\{|stq|,|qf/s|\} < 1$.

Proof. Recalling Ramanujan's identity (1.17) , then applying the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)$ on both sides of the above equation with respect to the parameter s, we get

$$\int_{-\infty}^{\infty}\frac{e^{-x^2+2mx}}{(tq^{1/2}e^{-2ikx};q)_\infty}\ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\left\{\frac{1}{(sq^{1/2}e^{2ikx};q)_\infty}\right\}dx$$
$$= \sqrt{\pi}e^{m^2}(-tqe^{-2mki};q)_\infty\ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)\left\{\frac{(-sqe^{2mki};q)_\infty}{(stq;q)_\infty}\right\}.$$

Now by using equations (2.3) and (2.2) the proof is complete. ■

4.3 Extension The Ramanujan's Beta Integral

Theorem 4.3 (Extension The Ramanujan's beta integral (1.17))

$$\int_0^\infty t^{x-1}\frac{(-yt;q)_\infty}{(t;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b;q)_{k+j}}{(d,e;q)_{k+j}}\frac{(-fq^{-x-j})^k}{(q;q)_k}\frac{(q^{x+1}/y;q)_j(-ftyq^{-x-1})^j}{(q;q)_j}q^{-\binom{j}{2}}d_qt$$
$$= \frac{\pi}{\sin(\pi x)}\frac{(q^{1-x};q)_\infty}{(q,yq^{-x};q)_\infty}\ {}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,-f\right). \tag{4.4}$$

Proof. Multiplying The Ramanujan's beta integral (1.17) by $(yq^{-x};q)_\infty$ we get

$$\int_0^\infty t^{x-1}\frac{1}{(t;q)_\infty}(-yt,yq^{-x};q)_\infty dt = \frac{\pi}{\sin(\pi x)}\frac{(q^{1-x};q)_\infty}{(q;q)_\infty}(y;q)_\infty.$$

Now applying the operator ${}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,f\theta\right)$ with respect to y, and using tow relations (2.5), (2.4) the proof is complete . ■

Setting $c = e$, $f = -d/ab$ in (4.4) and then using Hein's q-Gauss summation formula (1.8) we can obtain the following corollary:

Corollary 4.3.1. *We have*

$$\int_0^\infty t^{x-1}\frac{(-yt;q)_\infty}{(t;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b;q)_{k+j}}{(d;q)_{k+j}}\frac{(dq^{-x-j}/ab)^k}{(q;q)_k}\frac{(q^{x+1}/y;q)_j(dtyq^{-x-1}/ab)^j}{(q;q)_j}q^{-\binom{j}{2}}d_qt$$
$$= \frac{\pi}{\sin(\pi x)}\frac{(q^{1-x},y;q)_\infty}{(q,yq^{-x};q)_\infty}\frac{(d/a,d/b;q)_\infty}{(d,d/ab;q)_\infty}.$$

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