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Some Relations That Concerning Localization of Certain Types of Submodules

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Abstract

In this paper, we focus on localization of certain types of submodules such as pure submodules, idempotent submodules, multiplication and S −multiplication submodules and we try to obtain some relations between these submodules and their localizations. Also, we prove that under certain conditions certain properties of modules can be transferred from the modules to their localizations and conversely at multiplicative systems that isolate these modules and also isolate the rings on which these modules are defined.

1. Introduction

In 2004, M. M. Ali and D. J. Smith [1], have studied pure submodules of multiplication modules and obtained some properties of them and also they studied the relations of pure submodules with some other types of submodules. In 2010, L. H. Jahromi and A. Khaksari [6] have studied weakly pure submodules of multiplication modules, which are generalizations of pure submodules and they proved several properties of this type of submodules. Also, in 2011, A. Khaksari [7], has studied weakly pure submodules of multiplication modules, and in 2014, B. N. Shihab, H. Y. Khalaf and L. S. Mahmood [3], have studied purely and weakly purely cancellation modules and they proved some properties of each one and also obtained some relations between them. The purpose of this paper is to study the effect of localization on certain types of submodules such as pure submodules, almost pure submodules, locally pure submodules, idempotent submodules, multiplication and S −multiplication submodules and we try to organize the relations between them.

Throughout this paper, R is a commutative ring with identity and M is a left $R-$module, unless otherwise stated. Let $\emptyset \neq S \subseteq R$, then S is called a multiplicatively system if $0 \notin S$ and $a,b \in S$ implies that $ab \in S$ [8]. If S is a multiplicatively system in R, then we denote the localization of R at S by A_S (or $S^{-1}A$ [8]), which is $A_S = \{\frac{a}{s}: a \in A, s \in S\}$ [8]. If P is a prime ideal of R, then one can easily get that $R \setminus P$ is a multiplicatively system in R and in this case, we denote the localization of R at $R \setminus P$ by R_P, so that $A_P = \{\frac{a}{p}: a \in A, p \notin P\}$. A submodule K of M is called a pure submodule of M, if $AK = K \cap AM$, for every ideal A of R [2]. If N is a submodule of M, then $S_M(N) = \{r \in R: rx \in N$, for some $x \in M \setminus N\}$ and if A is an ideal of R, then $S_R(A) = \{r \in R: ra \in A$, for some $a \notin A\}$ and A is called a primal ideal of R, if $S_R(A)$, forms an ideal of R, which is always a prime ideal of R [4]. For a submodule K of M, $(K:M) = \{r \in R: rM \subseteq K\}$ and $Ann(M) = (0:M) = \{r \in R: rM = 0\}$. A submodule N of M is called an idempotent submodule of M if $N = [N:M]N$ [1] and it is called a multiplication submodule of M if $N \cap K = [K:N]N$ for every submodule K of M [1]. An ideal A of R is called a pure ideal of R if $A \cap B = AB$ for every ideal B of R **[3]**, equivalently A is a pure ideal of R if and only if $Aa = Ra$ for all $a \in A$ [1]. For a commutative ring R with identity, $J(R)$, is defined as the intersection of all maximal ideals of R [9].

2. The Main Results

Theorem 2.1. Let M be an R −module and N be a submodule of M. Then N is pure if and only if N_P is a pure submodule of M_P for every prime ideal P of R.

Proof. Let N be pure and P be a prime ideal of R. Let $\bar{A}$ be an ideal of R_P, then by [5, Proposition 2.16], $\bar{A} = A_P$ for some ideal A of R. As N is pure, we have $AN = N \cap AM$, then by [5, Corollary 2.3], we have $(AN)_P = A_P N_P$, so that we get $\bar{A}N_P = A_P N_P = (AN)_P = (N \cap AM)_P = N_P \cap (AM)_P = N_P \cap (A_P M_P) = N_P \cap (\bar{A}M_P)$, so that N_P is a pure submodule of M_P.

Conversely, let N be almost pure and A be any ideal of R. Then, for any prime ideal P of R and as N is almost pure, we have N_P is pure and as A_P is an ideal of R, we get $A_P N_P = N_P \cap A_P M_P$, this gives $(AN)_P = A_P N_P = N_P \cap A_P M_P = N_P \cap (AM)_P = (N \cap AM)_P$, so that $AN = N \cap AM$, so that N is a pure submodule of N.

Proposition 2.2. Let M be an R −module and N be a submodule of M. If N is almost pure, then N_P is a pure submodule of M_P for every maximal ideal P of R.

Proof. If P is a maximal ideal of R, then it is prime. By **Theorem 2.1**, we get N_P is a pure submodule of M_P.

Proposition 2.3. Let M be an R −module and N a submodule of M. If S is a multiplicative system in R such that $S \cap S_M(N) = \emptyset$, then $[N:M]_S = [N_S:M_S]$.

Proof. Let $\frac{r}{s} \in [N:M]_S$, where $r \in R, s \in S$, then $qr \in [N:M]$ for some $q \in S$, this gives $qrM \subseteq N$. Then we get $\frac{r}{s}M_S = \frac{q}{q}\frac{r}{s}M_S = \frac{qr}{qs}M_S = (qrM)_S \subseteq N_S$, which gives $\frac{r}{s} \in [N_S:M_S]$. Hence, we get $[N:M]_S \subseteq [N_S:M_S]$. Next, suppose that $\frac{r}{s} \in [N_S:M_S]$, then $\frac{r}{s}M_S \subseteq N_S$. Now, let $m \in M$, then $\frac{rm}{ss} = \frac{r}{s}\frac{m}{s} \in N_S$, so we get $qrm \in N$, for some $q \in S$. If $rm \notin N$, then we get $q \in S_M(N)$, so that $S \cap S_M(N) \neq \emptyset$, which is a contradiction, so that $rm \in N$, so we get $rM \subseteq N$, that is $r \in [N:M]$, this implies that $\frac{r}{s} \in [N:M]_S$. Hence $[N_S:M_S] \subseteq [N:M]_S$ and thus we have $[N:M]_S = [N_S:M_S]$.

Definition 2.4. Let M be an R −module and N a submodule of M. We call a prime ideal P of R not prime to N if $S_M(N) \subseteq P$ and we denote the set of all prime ideals P of R that are not prime to N by $\mathcal{S}_N^P = \{P: P$ is a prime ideal of R such that $S_M(N) \subseteq P\}$.

Corollary 2.5. Let M be an R −module and N a submodule of M, then $[N:M]_P = [N_P:M_P]$ for all $P \in \mathcal{S}_N^P$.

Proof. Let $P \in \mathcal{S}_N^P$, so that P is a prime ideal of R such that $S_M(N) \subseteq P$. Put $S = R\backslash P$, which is a multiplicative system in R and $S \cap S_M(N) \subseteq (R\backslash P) \cap P = \emptyset$, so that $S \cap S_M(N) = \emptyset$.

Hence, by Proposition 2.3, we get $[N:M]_P = [N:M]_S = [N_S:M_S] = [N_P:M_P]$.

Next, we prove that under certain conditions localization of idempotent submodules at multiplicative systems are also idempotent.

Proposition 2.6. Let M be an R −module, N a submodule of M and S a multiplicative system in R such that $S \cap S_M(N) = \emptyset$. If N is idempotent, then N_S is idempotent.

Proof. As N is idempotent, we have $N = [N:M]N$, then by Proposition 2.3, we get $N_S = ([N:M]N)_S = [N:M]_S N_S = [N_S:M_S]N_S$.

As a corollary to the above proposition we prove that localization of an idempotent submodule N at prime ideals which are not prime to N are also idempotent.

Corollary 2.7. Let M be an R −module. If N is an idempotent submodule of M, then N_P is an idempotent submodule of M_P for all $P \in \mathcal{S}_N^P$.

Proof. Let $P \in \mathcal{S}_N^P$, so that P is a prime ideal of R such that $S_M(N) \subseteq P$. Put $S = R\backslash P$, which is a multiplicative system in R and $S \cap S_M(N) \subseteq (R\backslash P) \cap P = \emptyset$, so that $S \cap S_M(N) = \emptyset$. Hence, by Proposition 2.6, we get N_P is an idempotent submodule of M_P.

In the following result, we show that for an R −module M and a multiplicative system S of R each submodule of M_S is a localization of a unique submodule of M.

Proposition 2.8. .Let M be an R −module and S a multiplicative system in R. If $\bar{K}$ is a submodule of M_S, then there exists a unique submodule K of M for which $\bar{K} = K_S$ and $S \cap S_M(K) = \emptyset$.

Proof. Let $s \in S$ be any element (this is possible since $S \neq \emptyset$) and $K = \{x \in M: \frac{x}{s} \in \bar{K}\}$. To show K is a submodule of M. As $0 \in M$ and $\frac{0}{s} \in \bar{K}$, so that $0 \in K$. Hence, $\emptyset \neq K \subseteq M$. Now, let $r \in R$ and $x, y \in K$, then $x, y \in M$ and $\frac{x}{s}, \frac{y}{s} \in \bar{K}$, then $\frac{x-y}{s} = \frac{x}{s} - \frac{y}{s} \in \bar{K}$. Also we have $\frac{r}{s} \in R_S$ and so $\frac{rx}{ss} = \frac{r}{s}\frac{x}{s} \in \bar{K}$, then $\frac{rx}{s} =$

$\frac{s}{s}\frac{s}{s}\frac{rx}{s} = \frac{ss}{s}\frac{rx}{ss} \in \bar{K}$, so that $rx \in K$. Hence K is a submodule of M. To show $\bar{K} = K_S$. Let $\frac{x}{t} \in \bar{K}$, where $x \in M, t \in S$, then $\frac{x}{s} = \frac{t}{t}\frac{x}{s} = \frac{t}{s}\frac{x}{t} \in \bar{K}$, so that $x \in K$ and then $\frac{x}{s} \in K_S$. Hence, $\bar{K} \subseteq K_S$. Next, let $\frac{x}{t} \in K_S$, where $x \in M, t \in S$, then $qx \in K$, for some $q \in S$, so that $\frac{qx}{s} \in \bar{K}$, then $\frac{x}{t} = \frac{s}{s}\frac{q}{q}\frac{x}{t} = \frac{s}{tq}\frac{qx}{s} \in \bar{K}$, so that $K_S \subseteq \bar{K}$. Hence $\bar{K} = K_S$. To show $S \cap S_M(K) = \emptyset$. If $S \cap S_M(K) \neq \emptyset$, then there exists $t \in S \cap S_M(K)$, so that $t \in S$ and $tx \in K$, for some $x \notin K$, this gives $\frac{tx}{s} \in \bar{K}$ and $\frac{x}{s} \notin \bar{K}$. On the other hand, we have $\frac{x}{s} = \frac{t}{t}\frac{x}{s} = \frac{1}{t}\frac{tx}{s} \in \bar{K}$, which is a contradiction, so that $S \cap S_M(K) = \emptyset$. Next, suppose that L is another submodule of M, for which $\bar{K} = L_S$ and $S \cap S_M(L) = \emptyset$. To show $L = K$. Then we have $L_S = K_S$. Let $x \in L$, then $\frac{x}{s} \in K_S = \bar{K}$, so that $x \in K$. Hence, $L \subseteq K$. Conversely, let $x \in K$, then $\frac{x}{s} \in \bar{K} = K_S = L_S$, so that $qx \in L$ for some $q \in S$. If $x \notin L$, then $q \in S_M(L)$ and this implies that $S \cap S_M(L) \neq \emptyset$ which is a contradiction, so that $x \in L$ and thus $K \subseteq L$. Hence $L = K = \{x \in M: \frac{x}{s} \in K_S = L_S\}$ and so the existence of a such submodule is unique.

Corollary 2.9. Let M be an R −module and P a prime ideal of R. If $\bar{K}$ is a submodule of M_P, then there exists a unique submodule K of M such that $\bar{K} = K_P$ and $S_M(K) \subseteq P$.

Proof. If we take $S = R \backslash P$, then S is a multiplicative system in R and since $S \cap S_M(K) = \emptyset$ if and only if $S_M(K) \subseteq P$, so the result follows directly from Proposition 2.8.

Remark 2.10. (1) Since $1 \in R \backslash P = S$, so one can take $s = 1$ in Proposition 2.8 and then the submodule K can be taken as $K = \{x \in M: \frac{x}{1} \in \bar{K}\}$.

(2) If we consider R as an R −module, then from Proposition 2.8 and Corollary 2.9, we get the following corollaries.

Corollary 2.11. Let R be a commutative ring with identity and S a multiplicative system in R. If $\bar{A}$ is an ideal of R_S, then there exists a unique ideal A of R for which $\bar{A} = A_S$ and $S \cap S_R(A) = \emptyset$.

Corollary 2.12. Let R be a commutative ring with identity and P a prime ideal of R. If $\bar{A}$ is an ideal of R_P, then there exists a unique ideal A of R for which $\bar{A} = A_P$ and $S_R(A) \subseteq P$.

Now we prove that localization of multiplication submodules of an R − module at multiplicative systems are also multiplication submodules.

Proposition 2.13. Let M be an R −module and S a multiplicative system in R. If N is a multiplication submodule of M, then N_S is a multiplication submodule of M_S.

Proof. Let $\bar{K}$ be any submodule of M_S, then by Proposition 2.8, $\bar{K} = K_S$ for the submodule $K = \{x \in M: \frac{x}{s} \in \bar{K}\}$ of M, where $s \in S$ and $S \cap S_M(K) = \emptyset$. Since N is a multiplication submodule of M, so $N \cap K = [K:N]N$, then by Proposition 2.3, we get $[K:N]_S = [K_S:N_S]$, so that $N_S \cap \bar{K} = N_S \cap K_S = (N \cap K)_S = ([K:N]N)_S = [K:N]_S N_S = [K_S:N_S]N_S = [\bar{K}:N_S]N_S$ and this means that N_S is a multiplication submodule of M_S.

As a corollary to the above result we prove that localization of submodules of an R −module at prime ideals are also multiplication submodules.

Corollary 2.14. Let M be an R −module and P a prime ideal of R. If N is a multiplication submodule of M, then N_P is a multiplication submodule of M_P.

Proof. The result follows directly by taking $S = R \backslash P$ in Proposition 2.13.

Now we introduce the following definitions.

Definition 2.15. Let M be an R −module. We define $\mathcal{S}^{J(R)} = \{N: N \leq M \text{ and } S_M(N) \subseteq J(R)\}$.

Definition 2.16. Let M be an R −module. We call a submodule N of M an $\mathcal{S}$ −multiplication submodule of M if $N \cap K = [K:N]N$ for every $K \in \mathcal{S}^{J(R)}$.

Now, we prove that locally multiplication submodules are $\mathcal{S}$ −multiplication.

Proposition 2.17. Let M be an R −module. If N is a submodule of M such that N_P is a multiplication submodule of M_P, for every prime ideal P of R, then N is an $\mathcal{S}$ −multiplication submodule of M.

Proof. Let P be any maximal ideal of R, so it is prime and hence N_P is a multiplication submodule of M_P. Let $K \in \mathcal{S}^{J(R)}$, that means $K \leq M$ and $S_M(K) \subseteq J(R) \subseteq P$, so that $P \in \mathcal{S}_K^P$. Then, K_P is a submodule of M_P and as N_P is a multiplication submodule of M_P, we get $N_P \cap K_P = [K_P:N_P]N_P$ and since $P \in \mathcal{S}_K^P$, so by Corollary 2.5, we get $(N \cap K)_P = N_P \cap K_P = [K_P:N_P]N_P = [K:N]_P N_P = ([K:N]N)_P$, so we get $N \cap K = [K:N]N$. Hence, N is an $\mathcal{S}$ −multiplication submodule of M.

Now we prove that localization of pure ideals of a ring at multiplicative systems are also pure.

Proposition 2.18. Let R be a commutative ring with identity and S a multiplicative system in R. If A is a pure ideal of R, then A_S is a pure ideal of R_S.

Proof. Let $\bar{B}$ be any ideal of R_S, so by Corollary 2.11, there exists a unique ideal B of R for which $\bar{B} = B_S$ and $S \cap S_R(B) = \emptyset$. As A is pure we have $A \cap B = AB$, which implies that $A_S \cap \bar{B} = A_S \cap B_S = (A \cap B)_S = (AB)_S = A_S B_S = A_S \bar{B}$. Hence, A_S is a pure ideal of R_S.

As a corollary to the above result we prove that localization of pure ideals of a ring at prime ideals are also pure.

Corollary 2.19. Let R be a commutative ring with identity and P a prime ideal of R. If A is a pure ideal of R, then A_P is a pure ideal of R_P.

Proof. The proof follows directly by putting $S = R \backslash P$ in Proposition 2.18.

Examples 2.20. (1) Consider the ring Z_6. Clearly $S = \{1, 5\}$ is a multiplicative system in Z_6. Then, we have $S_{Z_6}(0) = \{0, 2, 3, 4\}$, $S_{Z_6}(\{0, 2, 4\}) = \{0, 2, 4\}$, $S_{Z_6}(\{0, 3\}) = \{0, 3\}$ and $S_{Z_6}(Z_6) = \emptyset$ and it is clear that $S \cap S_{Z_6}(0) = \emptyset = S \cap S_{Z_6}(\{0, 2, 4\}) = S \cap S_{Z_6}(\{0, 3\}) = S \cap S_{Z_6}(Z_6)$, that means $S \cap S_{Z_6}(A) = \emptyset$ for every ideal A of Z_6. But, if we take the multiplicative system $S = \{1, 3\}$ in Z_6, then we see that for the ideal $A = \{0, 3\}$, we have $S_{Z_6}(\{0, 3\}) = \{0, 3\}$ and that $S \cap S_{Z_6}(A) = \{3\} \neq \emptyset$.

(2) Consider the ring of integers Z. Clearly $S = \{1\}$ and $S = \{-1,1\}$ are multiplicative systems in Z. For $S = \{1\}$, suppose that there exists an ideal A of Z for which $S \cap S_Z(A) \neq \emptyset$, then $1 \in S_Z(A)$, which implies that $1x \in A$ for some $x \notin A$ which is a contradiction, so that $\{1\} \cap S_Z(A) = \emptyset$ for every ideal A of Z. For $S = \{-1,1\}$, suppose that there exists an ideal A of Z for which $S \cap S_Z(A) \neq \emptyset$, then $-1 \in S_Z(A)$ or $1 \in S_Z(A)$ (or the both). If $-1 \in S_Z(A)$, then $-1x \in A$ for some $x \notin A$, then we get $-x \in A$, which gives $x \in A$ that is a contradiction and if $1 \in S_Z(A)$, then by using the same technique we get a contradiction. Hence $\{-1, 1\} \cap S_Z(A) = \emptyset$ for every ideal A of Z, that means for $S = \{1\}$ and $S = \{-1,1\}$, we have $S \cap S_Z(A) = \emptyset$ for every ideal A of Z.

(3) Consider the ring Z_{16}. The ideals of Z_{16} are $\{0\}$, $<2> = \{0,2,4,6,8,10,12,14\}$, $<4> = \{0,4,8,12\}$, $<8> = \{0,8\}$ and Z_{16}. Now we have $S_{Z_{16}}(0) = \{0, 2, 4, 6, 8, 10, 12, 14\} = <2>$, $S_{Z_{16}}(<2>) = <2>$, $S_{Z_{16}}(<4>) = \{0, 8\} = <8>$ and $S_{Z_{16}}(Z_{16}) = \emptyset$. Clearly we have $\{1, 3, 5, 7, 9, 11, 13, 15\} \cap \{0, 2, 4, 6, 8, 10, 12, 14\} = \emptyset$. Next, let S be any multiplicative system in Z_{16}, then $S \cap \{0, 2, 4, 6, 8, 10, 12, 14\} = \emptyset$, on the contrary suppose that $S \cap \{0, 2, 4, 6, 8, 10, 12, 14\} \neq \emptyset$, then we have one of the following cases:

(i) $0 \in S$, which is a contradiction, since $0 \notin S$.

(ii) $2 \in S$, then $0 = 2^4 \in S$ which is a contradiction, so that $2 \notin S$.

(iii) $4 \in S$, then $0 = 4^2 \in S$ which is a contradiction, so that $4 \notin S$.

(iv) $6 \in S$, then $0 = 6^4 \in S$ which is a contradiction, so that $6 \notin S$.

(v) $8 \in S$, then $0 = 8^2 \in S$ which is a contradiction, so that $8 \notin S$.

(vi) $10 \in S$, then $0 = 10^4 \in S$ which is a contradiction, so that $10 \notin S$.

(vii) $12 \in S$, then $0 = 12^2 \in S$ which is a contradiction, so that $12 \notin S$.

(vii) $14 \in S$, then $0 = 14^4 \in S$ which is a contradiction, so that $14 \notin S$.

Hence, $S \cap \{0, 2, 4, 6, 8, 10, 12, 14\} = \emptyset$ for every multiplicative system S in Z_{16} and this means that if S is any multiplicative system in Z_{16}, then $S \subseteq \{1, 3, 5, 7, 9, 11, 13, 15\}$. Next, we have $S \cap S_{Z_{16}}(0) \subseteq \{1, 3, 5, 7, 9, 11, 13, 15\} \cap \{0, 2, 4, 6, 8, 10, 12, 14\} = \emptyset$,
$S \cap S_{Z_{16}}(<2>) \subseteq \{1, 3, 5, 7, 9, 11, 13, 15\} \cap \{0, 2, 4, 6, 8, 10, 12, 14\} = \emptyset$,
$S \cap S_{Z_{16}}(<4>) \subseteq \{1, 3, 5, 7, 9, 11, 13, 15\} \cap \{0, 2, 4, 6, 8, 10, 12, 14\} = \emptyset$ and
$S \cap S_{Z_{16}}(Z_{16}) \subseteq \{1, 3, 5, 7, 9, 11, 13, 15\} \cap \emptyset = \emptyset$. Hence, we have $S \cap S_{Z_{16}}(A) = \emptyset$ for every multiplicative system S in Z_{16} and every ideal A of Z_{16}.

In view of the above examples we introduce the following definition.

Definition 2.21. Let R be a commutative ring with identity and M an R −module. If S is a multiplicative system in R, then we say S separates R (resp. M) if $S \cap S_R(A) = \emptyset$ (resp. $S \cap S_M(N) = \emptyset$) for every ideal A of R (resp. every submodule N of M) and we denote the set of all multiplicative systems in R that separate R by $\mathcal{S}_R = \{S : S$ is a multiplicative system separates $R\}$ and the set of all multiplicative systems in R that separate M by $\mathcal{S}_M = \{S : S$ is a multiplicative system separates $M\}$.

It is known that if P is a prime ideal of R, then $S = R\backslash P$ is a multiplicative system in R so in this case we make the following definition.

Definition 2.22. Let R be a commutative ring with identity and M an R −module. If P is a prime ideal of R. We say P isolates R (resp. M) if $S_R(A) \subseteq P$ (resp. $S_M(N) \subseteq P$) for every ideal A of R (resp. every submodule N of M) and we denote the set of all multiplicative systems in R that isolate R by $\mathcal{S}_R = \{R\backslash P: P$ is a prime ideal isolates $R\}$ and the set of all multiplicative systems in R that isolate M by $\mathcal{S}_M = \{R\backslash P: P$ is a prime ideal isolates $M\}$.

Note that, in the last definition if A is the prime ideal P of R, that is if $A = P$, then we have $S_R(A) = S_R(P) = P \subseteq P$, so the condition $S_R(A) \subseteq P$ is trivially satisfied when the ideal A is the prime ideal P itself.

It is known that, if the localizations of two ideals of a ring at a multiplicative system are equal then the ideals need not be equal as we see in the following example.

Example 2.23. Consider the ring Z_{12}. Take the ideals $A = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$ and $B = \{\bar{0}, \bar{4}, \bar{8}\}$ of Z_{12}. Now let $S = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}\}$ which is a multiplicative system of Z_{12}. It is easy to check that $A_S = \left\{\frac{\bar{0}}{\bar{1}}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}\right\} = B_S$, but clearly $A \neq B$.

Now, we prove that if the localizations of two ideals of a ring at a certain type of multiplicative systems are equal (in fact that multiplicative systems which isolate R), then the ideals are also equal.

Proposition 2.24. Let R be a commutative ring with identity. Let A and B be two ideals of R. If $S \in \mathcal{S}_R$ and $A_S = B_S$, then $A = B$. In particular, if $A_S = 0$, then $A = 0$.

Proof. Since $S \in \mathcal{S}_R$, so that S separates R, then $S \cap S_R(A) = \emptyset = S \cap S_R(B)$. As $S \neq \emptyset$, let $s \in S$. Let $a \in A$, then $\frac{a}{s} \in B_S$, then $ta \in B$, for some $t \in S$. If $a \notin B$, then $t \in S_R(B)$, this implies that $S \cap S_R(B) \neq \emptyset$, which is a contradiction, so that $a \in B$. Hence $A \subseteq B$ and as $S \cap S_R(A) = \emptyset$, by the same technique we get $B \subseteq A$. Hence, $A = B$. For the second part, let $a \in A$, then $\frac{a}{s} \in A_S = 0$, so that $ta = 0$ for some $t \in S$. If $a \neq 0$, then $t \in S_R(0)$, so that $S \cap S_R(0) \neq \emptyset$, which is a contradiction, so that $a = 0$. Hence, $A = 0$.

Corollary 2.25. Let R be a commutative ring with identity. Let A and B be two ideals of R. If $P \in \mathcal{S}_R$ and $A_P = B_P$, then $A = B$. In particular, if $A_P = 0$, then $A = 0$.

Proof. The proof follows directly by taking $S = R\backslash P$ in Proposition 2.24.

As in the case of rings we prove the same results of Proposition 2.24 and Corollary 2.25, for the modules.

Proposition 2.26. Let M be an R −module. Let N and K be two submodules of M. If $S \in \mathcal{S}_M$ and $N_S = K_S$, then $N = K$. In particular, if $N_S = 0$, then $N = 0$.

Proof. Since $S \in \mathcal{S}_M$, so that S separates M, then we have $S \cap S_M(N) = \emptyset = S \cap S_M(K)$. As $S \neq \emptyset$, let $s \in S$. Let $a \in N$, then $\frac{a}{s} \in K_S$, then $ta \in K$, for some $t \in S$. If $a \notin K$, then $t \in S_M(K)$, this implies that $S \cap S_M(K) \neq \emptyset$, which is a contradiction, so that $a \in K$. Hence $N \subseteq K$ and as $S \cap S_M(N) = \emptyset$, by the same technique we get $K \subseteq N$. Hence, $N = K$. For the second part, let $a \in N$, then $\frac{x}{s} \in N_S = 0$, so that $tx = 0$ for some $t \in S$. If $x \neq 0$, then $t \in S_M(0)$, so that $S \cap S_M(0) \neq \emptyset$, which is a contradiction, so that $x = 0$. Hence, $N = 0$.

Corollary 2.27. Let M be an R −module. Let N and K be two submodules of M. If $P \in \mathcal{S}_M$ and $N_P = K_P$, then $N = K$. In particular, if $N_P = 0$, then $N = 0$.

Proof. The proof follows directly by taking $S = R\backslash P$ in Proposition 2.26.

Now we prove that, for a ring R and a multiplicative system S of R, every pure ideal of R_S is a localization of a unique pure ideal of R.

Proposition 2.28. Let R be a commutative ring with identity and $S \in \mathcal{S}_R$. If $\bar{A}$ is a pure ideal of R_S, then there exists a unique pure ideal A of R such that $\bar{A} = A_S$.

Proof. By Corollary 2.11, there exists a unique ideal A of R such that $\bar{A} = A_S$ and $S \cap S_R(A) = \emptyset$. To show A is pure, let B be any ideal of R, then B_S is an ideal of R_S and as A_S is pure we have $A_S \cap B_S = A_S B_S$, this gives $(A \cap B)_S = (AB)_S$ and since $S \in \mathcal{S}_R$, so by Proposition 2.24, we get $A \cap B = AB$. Hence, A is a pure ideal of R.

Corollary 2.29. Let R be a commutative ring with identity and $P \in \mathcal{S}_R$. If $\bar{A}$ is a pure ideal of R_P, then there exists a unique pure ideal A of R such that $\bar{A} = A_P$.

Proof. The proof follows directly by taking $S = R\backslash P$ in Proposition 2.28.

Next we prove that, for an $R-$module M and a multiplicative system S which isolates M, each idempotent submodule of M_S is a localization of a unique idempotent submodule of M.

Proposition 2.30. Let M be an $R-$module and $S \in \mathcal{S}_M$. If $\bar{N}$ is an idempotent submodule of M_S, then there exists a unique idempotent submodule N of M such that $\bar{N} = N_S$.

Proof. By Corollary 2.8, there exists a unique submodule N of M such that $\bar{N} = N_S$ and $S \cap S_M(N) = \emptyset$. To show N is idempotent. By using Proposition 2.3, we have $[\bar{N}: M_S] = [N_S:M_S] = [N:M]_S$. As $\bar{N}$ is idempotent, we have $N_S = \bar{N} = [\bar{N}: M_S]\bar{N} = [N:M]_S N_S = ([N:M]N)_S$, then as $S \in \mathcal{S}_M$, by Proposition 2.27, we get $N = [N:M]N$. Hence, N is an idempotent submodule of M.

Corollary 2.31. Let M be an $R-$module and $P \in \mathcal{S}_M$. If $\bar{N}$ is an idempotent submodule of M_P, then there exists a unique idempotent submodule N of M such that $\bar{N} = N_P$.

Proof. The proof follows directly by taking $S = R\backslash P$ in Proposition 2.30.

Now we prove that, for an $R-$module M and a multiplicative system S in R that isolates M, every pure submodule of M_S is a localization of a unique pure submodule of M.

Proposition 2.32. Let M be an $R-$module and $S \in \mathcal{S}_M$. If $\bar{N}$ is a pure submodule of M_S, then there exists a unique pure submodule N of M such that $\bar{N} = N_S$.

Proof. By Corollary 2.8, there exists a unique submodule N of M such that $\bar{N} = N_S$ and $S \cap S_M(N) = \emptyset$. To show N is pure. If A is any ideal of R, then A_S is an ideal of R_S and as N_S is pure we get $A_S N_S = N_S \cap A_S M_S$, this implies that $(AN)_S = (N \cap AM)_S$. Since $S \in \mathcal{S}_M$, by Proposition 2.26, we get $AN = N \cap AM$. Hence N is a pure submodule of M.

Corollary 2.33. Let M be an $R-$module and $P \in \mathcal{S}_M$. If $\bar{N}$ is a pure submodule of M_P, then there exists a unique pure submodule N of M such that $\bar{N} = N_P$.

Proof. The proof follows directly by taking $S = R\backslash P$ in Proposition 2.32.

Proposition 2.34. Let M be an $R-$module and $S \in \mathcal{S}_M$. If $ann(M) = 0$, then $ann(M_S) = 0$.

Proof. Let $\frac{r}{s} \in ann(M_S)$, where $r \in R, s \in S$. Then, $\frac{r}{s}M_S = 0$ and by [5, Corollary 2.9], we have $\frac{r}{s}M_S = (rM)_S$, this gives $(rM)_S = 0$, and by Proposition 2.26, we get $rM = 0$, so that $r \in ann(M) = 0$, that is $r = 0$, which gives $\frac{r}{s} = 0$. Hence, $ann(M_S) = 0$.

Corollary 2.35. Let M be an $R-$module and $P \in \mathcal{S}_M$. If $ann(M) = 0$, then $ann(M_P) = 0$.

Proof. By taking $S = R\backslash P$ in Proposition 2.34, the proof follows directly.

Proposition 2.36. Let M be an $R-$module and $S \in \mathcal{S}_R$. If $ann(M_S) = 0$, then $ann(M) = 0$.

Proof. As $S \neq \emptyset$, let $s \in S$. Let $r \in ann(M)$, then $rM = 0$, this gives $(rM)_S = 0$ and by [5, Corollary 2.9], we have $\frac{r}{s}M_S = (rM)_S$, so that $\frac{r}{s}M_S = 0$. Hence $\frac{r}{s} \in ann(M_S) = 0$, so that $\frac{r}{s} = 0$. As $S \cap S_R(0) = \emptyset$, we get $r = 0$, so that $ann(M) = 0$.

Corollary 2.37. Let M be an $R-$module and $P \in \mathcal{S}_R$. If $ann(M_P) = 0$, then $ann(M) = 0$.

Proof. By taking $S = R\backslash P$ in Proposition 2.36, the proof follows directly.

References

[1] Ali, M. M. and Smith, D. J.: Pure submodules of Multiplication Modules, Contributions to Algebra and Geometry, Vol. 45 (1) (2004), 61-74.

[2] Anderson, F. W. and Fuller, K. R.: Rings and categories of modules, springer-verlarge, 1974.

[3] Bothaynah N. S., Hatam Y. K. and Layla S. M.: Purely and Weakly Purely Cancellation Modules, American Journal of Mathematics and Statistics, Vol. 4 (4) (2014), 186-190.

[4] Darani, A. Y. : Almost Primal Ideals in Commutative Rings, Chiang Mai J. Sci. Vol. 38(2) (2011), 161-165.

[5] Jabbar, A. K. : A generalization of prime and weakly prime submodules, Pure Mathematical sciences, Vol. 2 (1), (2013), 1-11.

[6] Jahromi, L. H. and Khaksari, A. : Weakly Pure Submodules of Multiplication Modules, Journal of Mathematical Sciences: Advances and Applications, Vol. 6 (2), (2010), 215-218.

[7] Khakasari, A. : Weakly Pure Submodules of Multiplication Modules, International Journal of Algebra, Vol. 5 (5), (2011), 247-250.

[8] Larsen, M. D. and McCarthy, P. J.: Multiplicative theory of ideals, Academic Press, New York and London, 1971.

[9] Wisbauer, R.: Foundations of module and ring theory, Gordon and Breach Science Publishers, 1991.