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On P −Weak Cancellation, P −Purely Cancellation and P −Weak Purely Cancellation Modules

Adil Kadir Jabbar, Dana Jamal Saeed

[1]Department of Mathematics, College of Science, University of Sulaimani, Sulaimani-Iraq

Abstract

In this paper, we introduce some new types of cancellation modules such as $P-$weak cancellation modules, $P-$purely cancellation modules and $P-$weak purely cancellation modules, where P is a prime ideal of a commutative ring R and M is an $R-$module. We try to give some equivalent conditions for each type and determine some relations between these types and some other types of modules such as weak cancellation modules, purely cancellation modules and weak purely cancellation modules.

Keywords: P −Weak cancellation modules, pure ideals, $P-$purely cancellation modules, P −weak purely cancellation modules, multiplicative systems and localization.

1. Introduction

In **[9]**, Naoum, A. G. and Mijbas, A. S. studied cancellation and weak cancellation modules and they obtained some properties of them. Also, they studied the relations of weak cancellation modules with some other types of modules such as projective modules and flat modules and they gave some conditions under which projective modules and flat modules are weak cancellation modules. In **[3]**, Bothainah, N. S., Hatam Y. Khalaf and Mahmood, L. S. purely and weakly purely cancellation modules and they constructed some equivalent conditions for each type. In **[8]**, Mahmood, L. S., Bothainah N. S. and Tahir, S. Rashid studied relatively cancellation modules and they obtained some relations of this type of modules with cancellation modules. Also they studied the effect of localization and trace of modules on this type of modules.

In this paper, we define some new types of cancellation modules such as $P-$ weak cancellation modules, $P-$purely cancellation modules and $P-$weak purely cancellation modules and we try to study the effect of localization on weak cancellation modules, purely cancellation modules and weak purely cancellation modules. Also we try to obtain some properties of them and find some equivalent conditions for these new types of cancellation modules.

Throughout this paper, R is a commutative ring with identity and M is a left $R-$module, unless otherwise stated. A nonempty subset S of R is called a multiplicatively closed set if $a,b \in S$ implies that $ab \in S$ and a multiplicatively closed set is called a multiplicatively system if $0 \notin S$ **[7]**. If S is a multiplicatively system in R, then we denote the localization of R at S by R_S (or $S^{-1}R$ **[7]**), which is $R_S = \{\frac{r}{s}: r \in R, s \in S\}$ **[7]**. If P is a prime ideal of R, then one can easily get that $R \setminus P$ is a multiplicatively system in R and in this case, we denote the localization of R at $R \setminus P$ by R_P, so that $R_P = \{\frac{r}{p}: r \in R, p \notin P\}$. If A is an ideal of R, then $A_S = \{\frac{a}{s}: a \in A, s \in S\}$ and if P is a prime ideal of R, then $A_P = \{\frac{a}{p}: a \in A, p \notin P\}$. If N is a submodule of M, then $S_M(N) = \{r \in R: rx \in N$, for some $x \in M \setminus N\}$ **[2]** and if A is an ideal of R, then $S_R(A) = \{r \in R: ra \in A$, for some $a \notin A\}$ **[4]**. For a submodule K of M, $(K:M) = \{r \in R: rM \subseteq K\}$ and $ann(M) = (0:M) = \{r \in R: rM = 0\}$. An ideal A of R is called a pure ideal of R if $A \cap B = AB$ for every ideal B of R **[3]**, equivalently A is a pure ideal of R if and only if $Aa = Ra$ for all $a \in A$ **[1]** and A is called a purely cancellation ideal if B is a pure ideal of R and C is any ideal of R such that $BA = CA$, then $B = C$ **[3]**. M is called a cancellation (weak cancellation) module if A and B are ideals of R such that $AM = BM$ implies $A = B$ ($A + ann(M) = B + ann(M)$) **[9]** and also M is called purely (weak purely) cancellation module if A is a pure ideal of R and B is an ideal o R such that $AM = BM$, then $A = B$ ($A + ann(M) = B + ann(M)$) **[3]**. M is called relatively (weak relatively) cancellation module if A is a prime ideal of R and B is an ideal o R such that $AM = BM$, then $A = B$ ($A + ann(M) = B + ann(M)$) **[8]**. The Jacobson radical of M, denoted by $J(M)$, is defined as $J(M) = \cap_K K$ is a maximal submodule of M and if R is considered as $R-$module then the Jacobson radical of R, denoted by $J(R)$, is defined by $J(R) = \cap_P P$ is a maximal ideal of R **[10]**.

2. $P-$Weak Cancellation Modules

First, we prove that the localization of the annihilator of an $R-$module is the same as the annihilator of the localization of the module.

Proposition 2.1. Let M be an $R-$module and P a prime ideal of R such that $S_M(0) \subseteq P$, then $(ann(M))_P = ann(M_P)$.

Proof. Let $\frac{r}{p} \in (ann(M))_P$, for $r \in R$ and $p \notin P$. Then $qr \in ann(M)$, for some $q \notin P$, which gives $qrM = 0$. Now let $\frac{x}{t} \in M_P$, where $x \in M, t \notin P$. We have $\frac{r}{p}\frac{x}{t} = \frac{q}{q}\frac{r}{p}\frac{x}{t} = \frac{qrx}{qpt} = 0$, so that $\frac{r}{p} \in ann(M_P)$, so that $(ann(M))_P \subseteq ann(M_P)$. Next, let $\frac{r}{p} \in ann(M_P)$, for $r \in R$ and $p \notin P$, so that $\frac{r}{p}M_P = 0$, then by **[5, Corollary 2.9]**, we get $(rM)_P = \frac{r}{p}M_P = 0$. If $x \in M$, then $\frac{rx}{1} = 0$, this gives $trx = 0$ for some $t \notin P$. Now, if $rx \neq 0$, then $t \in S_M(0) \subseteq P$, which is a contradiction, so that $rx = 0$, that is $rM = 0$, which means $r \in ann(M)$, so that $\frac{r}{p} \in (ann(M))_P$, thus we get $ann(M_P) \subseteq (ann(M))_P$. Hence $(ann(M))_P = ann(M_P)$.

Now, we introduce the definitions of $P-$cancellation and $P-$weak cancellation modules.

Definition 2.2. Let M be an $R-$module and P be a prime ideal of R. We call M a $P-$cancellation ($P-$weak cancellation) module if M_P is a cancellation (weak cancellation) module.

In the following we give an example of a $R-$module M in which $S \cap S_M(N) = \emptyset$, for each multiplicative system S in R, also it shows that for some prime ideal P of R, we have $S_M(N) \subseteq P$, for every submodule N of M.

Example 2.3. Consider the Z_{27} as $Z_{27}-$module. The submodules of Z_{27} are $<\overline{0}>, <\overline{3}>, <\overline{9}>$ and Z_{27}. Now, it is easy to check that:

$S_{Z_{27}}(<\overline{0}>) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}\}$.

$S_{Z_{27}}(<\overline{3}>) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}\}$.

$S_{Z_{27}}(<\overline{9}>) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}\}$.

$S_{Z_{27}}(Z_{27}) = \emptyset$.

Next, if S is any multiplicative system in Z_{27}, then S does not contain any multiple of $\overline{3}$, since if $s = \overline{3}\overline{x} \in S$, then we get $0 = s^3 \in S$, which is a contradiction, so that we get $S \subseteq \{\overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{7}, \overline{8}, \overline{10}, \overline{11}, \overline{13}, \overline{14}, \overline{16}, \overline{17}, \overline{19}, \overline{20}, \overline{22}, \overline{23}, \overline{25}, \overline{26}\}$. Clearly we have $S \cap S_{Z_{27}}(<\overline{0}>) = \emptyset = S \cap S_{Z_{27}}(<\overline{3}>) = S \cap S_{Z_{27}}(<\overline{9}>) = S \cap S_{Z_{27}}(Z_{27})$.

That means, $\cap S_{Z_{27}}(N) = \emptyset$, for every multiplicative system S in Z_{27} and for every submodule N of Z_{27}. On the other hand we have $P = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}\}$ is a prime ideal of Z_{27} and clearly we have $S_{Z_{27}}(N) = P \subseteq P$, for every submodule $N \neq Z_{27}$ and $S_{Z_{27}}(Z_{27}) = \emptyset \subseteq P$, that is we have $S_{Z_{27}}(N) \subseteq P$, for every submodule N of Z_{27}.

Remark 2.4. Let R be a commutative ring with identity and S is a multiplicative system in R such that $S \cap S_R(I) = \emptyset$, for every ideal I of R, then it is easy to prove that:

(1) If A is a pure ideal of R, then A_S is a pure ideal of R_S.

(2) If $\overline{A}$ is a pure ideal of R_S, then there exists a unique pure ideal A of R with $\overline{A} = A_S$.

If P is a prime ideal of R, then $S = R \backslash P$ is a multiplicative system in R. In view of this the above remark becomes as follows:

Let R be a commutative ring with identity and P is a prime ideal of R such that $S_R(I) \subseteq P$, for every ideal I of R, then we have:

(1) If A is a pure ideal of R, then A_P is a pure ideal of R_P.
(2) If $\overline{A}$ is a pure ideal of R_P, then there exists a unique pure ideal A of R with $\overline{A} = A_P$.

It is known that, if M is a cancellation $R-$module and A is an ideal of R, then AM is a purely cancellation module if and only if A is a purely cancellation ideal of R **[3, Proposition 2.10]**.

Proposition 2.5. Let M be an $R-$module and P be a prime ideal of R such that $S_M(K) \subseteq P$ for every submodule K of M. If N is a purely cancellation submodule of M, then N_P is a purely cancellation submodule of M_P.

Proof. To show N_P is a purely cancellation submodule of M_P. Let $\overline{A}N_P = \overline{B}N_P$, where $\overline{A}$ is a pure ideal of R_P and $\overline{B}$ is any ideal of R_P. Then by **Remark 2.4**, $\overline{A} = A_P$, where A is a pure ideal of R and $\overline{B} = B_P$, where B is an ideal of R. Then we get $(AN)_P = A_P N_P = \overline{A}N_P = \overline{B}N_P = B_P N_P = (BN)_P$. As $S_M(AN) \subseteq P$ and $S_M(BN) \subseteq P$, we get $AN = BN$ and as N is a purely cancellation module, we get $A = B$ and thus $\overline{A} = A_P = B_P = \overline{B}$. Hence N_P is a purely cancellation submodule of M_P.

In the next result we show that under certain condition the localization of a purely cancellation ideal of a ring is also purely cancellation.

Proposition 2.6. Let R be a commutative ring with identity and P be a prime ideal of R such that $S_R(I) \subseteq P$ for every ideal I of R. If A is a purely cancellation ideal of R, then A_P is a purely cancellation ideal of R_P.

Proof. To show A_P is a purely cancellation ideal of R_P. Let $\overline{B}A_P = \overline{C}A_P$, where $\overline{B}$ is a pure ideal of R_P and $\overline{C}$ is any ideal of R_P. Then by **Remark 2.4**, $\overline{B} = B_P$, where B is a pure ideal of R and $\overline{C} = C_P$, where C is an ideal of R. Then we get $(BA)_P = B_P A_P = \overline{B}A_P = \overline{C}A_P = C_P A_P = (CA)_P$. As $S_M(BA) \subseteq P$ and $S_M(CA) \subseteq P$, we get $BA = CA$ and as A is a purely cancellation ideal, we get $B = C$ and thus $\overline{B} = B_P = C_P = \overline{C}$. Hence A_P is a purely cancellation ideal of R_P.

Next, we extend a property of cancellation modules to $P-$cancellation modules.

Proposition 2.7. Let M be an $R-$module and P be a prime ideal of R such that $S_M(K) \subseteq P$ for every submodule K of M and $S_R(I) \subseteq P$ for every ideal I of R and let A be an ideal of R. If M is a $P-$cancellation module, then AM is a purely cancellation module if and only if A is a purely cancellation ideal.

Proof. $(\Rightarrow)$ Since M is a $P-$cancellation module, so M_P is a cancellation module. Let AM be a purely cancellation module, then by **Proposition 2.6**, we get $(AM)_P = A_P M_P$ is a purely cancellation module, so that by **[3, Proposition 1.10]**, A_P is a purely cancellation ideal of R_P. To show A is a purely cancellation ideal of R. Let B be a pure ideal of R and C is an ideal of R such that $BA = CA$, then we get $B_P A_P = (BA)_P = (CA)_P = C_P A_P$. Now, by **Remark 2.4**, we get B_P is a pure ideal of R_P and as A_P is an ideal of R_P and M_P is a cancellation module, we get $B_P = C_P$ and since $S_R(B) \subseteq P$ and $S_R(C) \subseteq P$, we get $B = C$. Hence A is a purely cancellation ideal of R.

$(\Leftarrow)$ Let A be a purely cancellation ideal of R, so by **Proposition 2.6**, we get A_P is a purely cancellation ideal of R_P and as M_P is a cancellation module by **[3, Proposition 1.10]**, we have $A_P M_P$ is a purely cancellation module. To show AM is a purely cancellation module, so let $BAM = CAM$, where B is a pure ideal of R and C is an ideal of R, then B_P is a pure ideal of R_P and C_P is an ideal of R_P. Now, we have $B_P A_P M_P = (BAM)_P = (CAM)_P = C_P A_P M_P$ and as $A_P M_P$ is a purely cancellation module, so we get $B_P = C_P$. As $S_R(B) \subseteq P$ and $S_R(C) \subseteq P$, we get $B = C$. Hence AM is a purely cancellation module.

Next, we prove the following result which will be used to prove some other results.

Lemma 2.8. Let M be an $R-$module. If A, B are ideals of R and N a submodule of M, then
(1) $S_M(N:M) \subseteq S_M(N)$.
(2) $S_R(A:B) \subseteq S_R(A)$.

Proof. (1) Let $r \in S_M(N:M)$, then $rx \in (N:M)$ for some $x \notin (N:M)$, this implies that $rxM \subseteq N$ and $xM \nsubseteq N$, so there exists $m \in M$ for which $xm \notin N$ but then $rxm \in N$. Hence $r \in S_M(N)$, thus $S_M(N:M) \subseteq S_M(N)$.

(2) Let $r \in S_R(A:B)$, then $rx \in (A:B)$ for some $x \notin (A:B)$, this implies that $rxB \subseteq A$ and $xB \nsubseteq A$, so there exists $m \in B$ for which $xm \notin A$ but then $rxm \in A$. Hence $r \in S_R(A)$, thus $S_R(A:B) \subseteq S_R(A)$.

Next, we give some equivalent conditions for an $R-$module to be a $P-$weak cancellation module.

Proposition 2.9. If M is an $R-$module and P is a prime ideal of R. If $S_M(I + ann(M)) \subseteq P$ and $S_M(IM) \subseteq P$, for every ideal I of R, then the following conditions are equivalent:

(1) M is a P −weak cancellation module.
(2) For ideals A, B of R, $AM \subseteq BM$ implies $A \subseteq B + ann(M)$.
(3) If for $a \in R$ and ideal B of R, $(a)M \subseteq BM$ implies that $a \in B + ann(M)$.
(4) $(AM : M) = A + ann(M)$ for every ideal A of R.
(5) $(A + ann(M) : B) = (AM : BM)$ for any two ideals A and B of R.

Proof. (1) ⇒ (2). Let M_P be a weak cancellation module and A, B are ideals of R such that $AM \subseteq BM$. Then A_P and B_P are ideals of R_P and $A_P M_P = (AM)_P \subseteq (BM)_P = B_P M_P$. As M_P is weak cancellation, by **[9, Proposition 1.4]**, we get $A_P \subseteq B_P + ann(M_P)$ and as $S_M(0) = S_M(0M) \subseteq P$, by **Proposition 2.1**, we get $A_P \subseteq B_P + ann(M_P) = (B + ann(M))_P$. Let $a \in A$, then $\frac{a}{1} = \frac{b+s}{p}$, for some $b \in B$, $s \in ann(M)$ and $p \notin P$. Then $qpa = qb + qs \in B + ann(M)$, for some $q \notin P$. If $a \notin B + ann(M)$, then $qp \in S_M(B + ann(M)) \subseteq P$, this gives $q \in P$ or $p \in P$, which is a contradiction, so that $a \in B + ann(M)$. Hence $A \subseteq B + ann(M)$.

(2) ⇒ (3) ⇒ (4) ⇒ (5). The proof follows directly by **[9, Proposition 1.4]**.

(5) ⇒ (1). Let $\overline{A}$ and $\overline{B}$ be ideals of R_P, then there exist ideals A and B of R such that $\overline{A} = A_P$ and $\overline{B} = B_P$. So that by the given condition we have $(A + ann(M) : B) = (AM : BM)$, which gives that $(A + ann(M) : B)_P = (AM : BM)_P$ and since we have $S_R(A + ann(M) : B) \subseteq S_R(A + ann(M)) \subseteq P$ and $S_M(AM : BM) \subseteq S_M(AM) \subseteq P$ and $S_M(0) = S_M(0M) \subseteq P$, so we get $(A_P + ann(M_P) : B_P) = (A_P M_P : B_P M_P)$, that is $(\overline{A} + ann(M_P) : \overline{B}) = (\overline{A} M_P : \overline{B} M_P)$, so by **[9, Proposition 1.4]**, we get M_P is a weak cancellation module that means M is a P −weak cancellation module.

By combining **Proposition 2.9** and **[9, Proposition 1.4]**, we get the following theorem.

Theorem 2.10. If M is an R −module and P is a prime ideal of R. If $S_M(I + ann(M)) \subseteq P$ and $S_M(IM) \subseteq P$, for every ideal I of R, then the following conditions are equivalent:
(1) M is a weak cancellation module.
(2) M is a P −weak cancellation module.
(3) For ideals A, B of R, $AM \subseteq BM$ implies $A \subseteq B + ann(M)$.
(4) If for $a \in R$ and ideal B of R, $(a)M \subseteq BM$ implies that $a \in B + ann(M)$.
(5) $(AM : M) = A + ann(M)$ for every ideal A of R.
(6) $(A + ann(M) : B) = (AM : BM)$ for any two ideals A and B of R.

It is known that, if an R −module M is a cancellation module then $M_P \neq 0$ for each maximal ideal P of R **[9, Proposition 2.1]**. Here we show that under certain condition this property is also true for every R −module M.

Proposition 2.11. Let M be an R −module and P is a maximal ideal of R such that $S_M(0) \subseteq P$, then $M_P \neq 0$.

Proof. If possible suppose that $M_P = 0$, then for any $m \in M$, we have $\frac{m}{1} = 0$, this implies $qm = 0$ for some $q \notin P$. If $m \neq 0$, then $q \in S_M(0) \subseteq P$, that is a contradiction, so that $m = 0$ which gives that $M = 0$ which is again a contradiction. Hence $M_P \neq 0$.

3. P −purely cancellation modules.

We introduce the following definition.

Definition 3.1. Let M be an R −module and P be a prime ideal of R. We call M a P −purely cancellation module if M_P is a purely cancellation module.

Now, we give some equivalent conditions for an R −module to be a P −purely cancellation module.

Proposition 3.2. Let M be an R −module and P be a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for every ideal I of R and every submodule N of M, then the following conditions are equivalent.
(1) M is a P −purely cancellation module.
(2) If $AM \subseteq BM$, where A is any ideal of R and B is a pure ideal of R, then $A \subseteq B$.
(3) If $(a)M \subseteq BM$, where $a \in R$ and B is a pure ideal of R, then $a \in B$.
(4) $(AM : M) = A$ for all pure ideals A of R.
(5) $(AM : BM) = (A : B)$ for all ideals B of R and for all pure ideals A of R.

Proof. (1) ⇒ (2). Let M be a P −purely cancellation module and let $AM \subseteq BM$, where A is any ideal of R and B is a pure ideal of R. To show $A \subseteq B$. We have M_P is a purely cancellation module. Then A_P is an

ideal of R_P and as B is pure, by **Remark 242**, we get B_P is a pure ideal of R_P and also we have $A_P M_P = (AM)_P \subseteq (BM)_P = B_P M_P$, so by **[3, Theorem 1.5]**, we get $A_P \subseteq B_P$. As $S_R(B) \subseteq P$, we get $A \subseteq B$.
$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. The proof follows directly by **[3, Theorem 1.5]**.
$(5) \Rightarrow (1)$. Suppose that the condition (2) is satisfied. To show M is a $P-$purely cancellation module it is enough to show that M_P is a purely cancellation module. Let $\overline{A}$ be any pure ideal of R_P and $\overline{B}$ be any ideal of R_P, then by **Remark 2.4**, there exists a unique pure ideal A of R such that $\overline{A} = A_P$ and also there exists an ideal B of R such that $\overline{B} = B_P$. Now, by the given condition (2), we get $(AM{:}BM) = (A{:}B)$. As $S_M(AM) \subseteq P$, so that we get $(\overline{A}M_P{:}\overline{B}M_P) = (A_P M_P{:}B_P M_P) = (AM{:}BM)_P = (A{:}B)_P = (A_P{:}B_P) = (\overline{A}{:}\overline{B})$. Hence by **[2, Theorem 1.5]**, we get M_P is a purely cancellation module, so that M is a $P-$purely cancellation module.

By combining **Proposition 3.2** and **[3, Theorem 1.5]**, we get the following theorem.

Theorem 3.3. Let M be an $R-$module and P be a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for every ideal I of R and every submodule N of M, then the following conditions are equivalent.
(1) M is a purely cancellation module.
(2) M is a $P-$purely cancellation module.
(3) If $AM \subseteq BM$, where A is any ideal of R and B is a pure ideal of R, then $A \subseteq B$.
(4) If $(a)M \subseteq BM$, where $a \in R$ and B is a pure ideal of R, then $a \in B$.
(5) $(AM{:}M) = A$ for all pure ideals A of R.
(6) $(AM{:}BM) = (A{:}B)$ for all ideals B of R and for all pure ideals A of R.

4. $P-$Weak Purely Cancellation Modules

Now we introduce the following definition.

Definition 4.1. Let M be an $R-$module and P is a prime ideal of R. We call M a $P-$weak purely cancellation module if M_P is a weak cancellation module.

Theorem 4.2. Let M be an $R-$module and P be a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for every ideal I of R and every submodule N of M, then the following conditions are equivalent.
(1) M is a $P-$weak purely cancellation module.
(2) If A is an ideal of R and B is a pure ideal of R, then $AM \subseteq BM$ implies $A \subseteq B + ann(M)$.
(3) If $a \in R$ and B is a pure ideal of R, then $(a)M \subseteq BM$ implies $a \in B + ann(M)$.
(4) $(AM{:}M) = A + ann(M)$, for all pure ideals A of R.
(5) $(AM{:}BM) = (A + ann(M){:}M)$, where A is a pure ideal of R and B is any ideal of R.

Proof. The proof follows by using the same technique as we have used in **Proposition 3.2**.

5. $P-$Weak Relatively Cancellation Modules

We introduce the following definition.

Definition 5.1. Let M be an $R-$module and P is a prime ideal of R. We call M a $P-$relatively ($P-$weak relatively) cancellation module if M_P is a relatively (weak relatively) cancellation module.

It is known that, if P is a prime ideal of R, then R_P is a local ring with P_P as its unique maximal ideal and also we have $S_R(P) = P$ **[7]**. Now, by using this fact we prove the following result.

Proposition 5.2. Let M be an $R-$module and P is a prime ideal of R, such that $S_R(J(R)) \subseteq P$ and $S_M(0) \subseteq P$. If M is a $P-$relatively cancellation module, then $ann(M) \subseteq P$.

Proof. Since M is a $P-$relatively cancellation module, so that M_P is relatively cancellation module, so by **[8, Corollary 3.3]**, we get $ann(M_P) \subseteq J(R_P)$. Since $S_M(0) \subseteq P$, so we have $(ann(M))_P = ann(M_P)$, and as P_P is the unique maximal ideal of R_P, we have $J(R_P) = P_P$, so that we get $(ann(M))_P \subseteq P_P$. Now, let $x \in ann(M)$, then $\frac{x}{1} \in P_P$, so that $px \in P$, for some $p \notin P$ and as P is a prime ideal, we get $x \in P$, which gives $S_R(P) \subseteq P$, so that $ann(M) \subseteq P$.

The following result proves that under certain conditions the localization of a maximal submodule of an $R-$module is also maximal.

Proposition 5.3. Let M be an $R-$module and S is a multiplicatively system in R. If K is a maximal submodule of M such that $S \cap S_M(K) = \emptyset$, then K_S is a maximal submodule of M_S. Furthermore, if P is a prime ideal of R, then K_P is a maximal submodule of M_P.

Proof. Let $K_S = M_S$ and $s \in S$, then for each $m \in M$, we have $\frac{m}{s} \in K_S$, this gives that $tm \in K$, for some $t \in S$. If $m \notin K$, then we get $t \in S_M(K)$, thus we get $S \cap S_M(K) \neq \emptyset$, that is a contradiction, so that K_S is a proper ideal of M_S. Now, let $K_S \subseteq \overline{L} \subseteq M_S$, where $\overline{L}$ is a submodule of M_S. Then $\overline{L} = L_S$ for some submodule L of M with $S \cap S_M(L) = \emptyset$, so that $K_S \subseteq L_S \subseteq M_S$, and as $S \cap S_M(L) = \emptyset$, we get $K \subseteq L \subseteq M$, this gives $K_S = L_S$ or $L_S = M_S$, that is $K_S = \overline{L}$ or $\overline{L} = M_S$. Hence K_S is a maximal submodule of M_S. For the proof of the second part, as $S = R\backslash P$ is a multiplicatively system in R, so by taking $S = R\backslash P$ in the proof of the first part we get the result at once.

Next we prove that, for an $R-$module M and a multiplicative system S, each maximal submodule of M_S is a localization of a unique maximal submodule of M.

Proposition 5.4. Let M be an $R-$module and S is a multiplicatively system in R. If $\overline{K}$ is a maximal submodule of M_S, then there exists a maximal submodule K of S with $\overline{K} = K_S$ and $S \cap S_M(K) = \emptyset$. Furthermore, if P is a prime ideal of R, then there exists a maximal submodule K of S with $\overline{K} = K_P$ and $S_M(K) \subseteq P$.

Proof. As $\overline{K}$ is a submodule of M_S, we get $\overline{K} = K_S$, for some submodule K of M with $S \cap S_M(K) = \emptyset$. It remains to show that K is a maximal submodule of M. Let $K \subseteq L \subseteq M$, where L is any submodule of M. If $K = M$, then we get $K_S = M_S$, this contradicts the fact that K_S is a maximal submodule of M_S, so that $K \neq M$, that is K is a proper submodule of M. Also we have $K_S \subseteq L_S \subseteq M_S$ and since K_S is a maximal submodule of M_S, we get $K_S = L_S$ or $L_S = M_S$. Now, if $K_S = L_S$, then as $S \cap S_M(K) = \emptyset$, we get $L \subseteq K$ and hence $K = L$ and if $L_S = M_S$, then as $L = K$, we get $S_M(L) = S_M(K) = \emptyset$, so that $S \cap S_M(L) = S \cap S_M(K) = \emptyset$. Let $s \in S$ (this is possible since $S \neq \emptyset$). Now for any $m \in M$, we have $\frac{m}{s} \in L_S$, so we get $tm \in L$, for some $t \in S$ and if $m \notin L$, we get $t \in S_M(L)$, this gives that $S \cap S_M(L) \neq \emptyset$, that is a contradiction, so that we must have $m \in L$, this gives $L = M$. Hence K is a maximal submodule of M. The proof of the second part follows directly by putting $S = R\backslash P$ in the proof of the first part.

By combining **Proposition 5.3** and **Proposition 5.4** we get the following theorem which provides a one to one correspondence between the maximal submodules N of M that does not intersect $S_M(N)$ and the maximal submodules of M_S.

Theorem 5.5. Let M be an $R-$module and S is a multiplicatively system in R, then there is a one to one correspondence between the maximal submodules of M_S and the maximal submodules N of M for which $S \cap S_M(N) = \emptyset$. Furthermore, if P is a prime ideal of R, then there is a one to one correspondence between the maximal submodules of M_P and the maximal submodules N of M for which $S_M(N) \subseteq P$.

Proof. Let $\Gamma = \{\overline{N} : \overline{N}$ is a maximal submodule of $M_S\}$ and $\Psi = \{N : N$ is a maximal submodule of M for which $S \cap S_M(N) = \emptyset\}$. We define $f : \Psi \to \Gamma$ as follows: let $N \in \Psi$, so that N is a maximal submodule of M with $S \cap S_M(N) = \emptyset$, then by **Proposition 5.3**, we get N_S is a maximal submodule of M_S, so that $N_S \in \Gamma$ and we define $f(N) = N_S$. By using **Proposition 5.3** and **Proposition 5.4** one can easily prove that f is a bijective mapping so that f defines a one to one correspondence between Ψ and Γ. The proof of the second part follows directly by taking $S = R\backslash P$ as a multiplicative system.

By using the **Proposition 5.4**, we are able to prove the following result.

Proposition 5.6. Let M be an $R-$module and P is a prime ideal of R, then $(J(M))_P \subseteq J(M_P)$.

Proof. Let $\frac{m}{p} \in (J(M))_P$, where $m \in M, p \notin P$. Then $qm \in J(M)$, for some $q \notin P$. Now, let $\overline{N}$ be a maximal submodule of M_P, so by **Proposition 5.4**, there exists a maximal submodule N of S such that $\overline{N} = N_S$ and $S \cap S_M(N) = \emptyset$, then we get $qm \in N$. If $m \notin N$, then $q \in S_M(N)$, thus we get $S \cap S_M(N) \neq \emptyset$, that is a contradiction, so we get $m \in N$ and thus $\frac{m}{p} \in N_S = \overline{N}$. Hence $\frac{m}{p} \in J(M_P)$, so that $(J(M))_P \subseteq J(M_P)$.

In the following result we give some equivalent conditions for an $R-$module to be a $P-$relatively cancellation module.

Theorem 5.7. Let M be an $R-$module and P be a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for every ideal I of R and every submodule N of M, then the following conditions are equivalent.

(1) M is a $P-$relatively cancellation module.

(2) If A is an ideal of R and B is a prime ideal of R, then $AM \subseteq BM$ implies that $A \subseteq B$.

(3) If $a \in R$ and B is a prime ideal of R, then $(a)M \subseteq BM$ implies $a \in B$.

(4) $(AM:M) = A$, for all prime ideals A of R.
(5) $(AM:BM) = (A:B)$, where A is a prime ideal of R and B is any ideal of R.

Proof. $(1) \Rightarrow (2)$. Suppose that M is a P −relatively cancellation module and $AM \subseteq BM$, where A is an ideal of R and B is a prime ideal of R. Then we have M_P is a relatively cancellation module and A_P is an ideal of R_P and as $S_R(B) \subseteq P$, we get B_P is a prime ideal of R_P, so by **[8, Theorem 2.5]**, we have $A_P \subseteq B_P$ and as $S_M(0) \subseteq P$, we get $A \subseteq B$.
$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. The proof follows directly by **[8, Theorem 2.5]**.
$(5) \Rightarrow (1)$. Suppose that the condition (5) holds. To show M is a P −relatively cancellation module it is enough to show that M_P is a relatively cancellation module. Let $\overline{A}$ be any prime ideal of R_P and $\overline{B}$ be any ideal of R_P, then there exists a prime ideal A of R and an ideal B of R with $\overline{A} = A_P$ and $\overline{B} = B_P$. Then by the given condition we have $(AM:BM) = (A:B)$, then we have $(\overline{A}M_P:\overline{B}M_P) = (A_P M_P:B_P M_P) = (AM:BM)_P = (A:B)_P = (A_P:B_P) = (\overline{A}:\overline{B})$, so by **[8, Theorem 6.6]**, we get M_P is a relatively cancellation module and thus M is a P −relatively cancellation module.

By combining **Theorem 5.7** and **[8, Theorem 2.5]**, we get the following theorem.

Theorem 5.8. Let M be an R −module and P be a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for every ideal I of R and every submodule N of M, then the following conditions are equivalent.
(1) M is a relatively cancellation module.
(2) M is a P −relatively cancellation module.
(3) If A is an ideal of R and B is a prime ideal of R, then $AM \subseteq BM$ implies that $A \subseteq B$.
(4) If $a \in R$ and B is a prime ideal of R, then $(a)M \subseteq BM$ implies $a \in B$.
(5) $(AM:M) = A$, for all prime ideals A of R.
(6) $(AM:BM) = (A:B)$, where A is a prime ideal of R and B is any ideal of R.

Next we give some equivalent conditions for an R −module to be a P −weak relatively cancellation module.

Theorem 5.9. Let M be an R −module and P be a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for every ideal I of R and every submodule N of M, then the following conditions are equivalent.
(1) M is a P −weak relatively cancellation module.
(2) If A is an ideal of R and B is a prime ideal of R, then $AM \subseteq BM$ implies that $A \subseteq B + ann(M)$.
(3) If $a \in R$ and B is a prime ideal of R, then $(a)M \subseteq BM$ implies $a \in B + ann(M)$.
(4) $(AM:M) = A + ann(M)$, for all prime ideals A of R.
(5) $(AM:BM) = (A + ann(M):M)$, where A is a prime ideal of R and B is any ideal of R.

Proof. $(1) \Rightarrow (2)$. Suppose that M is a P −weak relatively cancellation module and $AM \subseteq BM$, where A is an ideal of R and B is a prime ideal of R. Then we have M_P is a weak relatively cancellation module and A_P is an ideal of R_P and as $S_R(B) \subseteq P$, we get B_P is a prime ideal of R_P, so by **[8, Theorem 6.6]**, we have $A_P \subseteq B_P + ann(M_P)$ and as $S_M(0) \subseteq P$, we get $(ann(M))_P = ann(M_P)$, so we get $A_P \subseteq (B + ann(M))_P$ and as $S_R(B + ann(M)) \subseteq P$, we get $A \subseteq B + ann(M)$.
$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. The proof follows directly by **[8, Theorem 6.6]**.
$(5) \Rightarrow (1)$. Suppose that the condition (2) holds. To show M is a P −weak relatively cancellation module it is enough to show that M_P is weak relatively cancellation module. Let $\overline{A}$ be any prime ideal of R_P and $\overline{B}$ be any ideal of R_P, then there exists a prime ideal A of R and an ideal B of R with $\overline{A} = A_P$ and $\overline{B} = B_P$. Then by the given condition we have $(AM:BM) = (A + ann(M):M)$, then we have $(\overline{A}M_P:\overline{B}M_P) = (A_P M_P:B_P M_P) = (AM:BM)_P = (A + ann(M):M)_P = (A_P + ann(M_P):M_P) = (\overline{A} + ann(M_P):M_P)$, so by **[8, Theorem 6.6]**, we get M_P is a weak relatively cancellation module and thus M is a P − weak relatively cancellation module.

By combining **Theorem 5.9** and **[8, Theorem 6.6]**, we get the following theorem.

Theorem 5.10. Let M be an R −module and P be a prime ideal of R such that $S_R(I) \subseteq P$ and $S_M(N) \subseteq P$, for every ideal I of R and every submodule N of M, then the following conditions are equivalent.
(1) M is a weak relatively cancellation module.
(2) M is a P −weak relatively cancellation module.
(3) If A is an ideal of R and B is a prime ideal of R, then $AM \subseteq BM$ implies that $A \subseteq B + ann(M)$.
(4) If $a \in R$ and B is a prime ideal of R, then $(a)M \subseteq BM$ implies $a \in B + ann(M)$.

(5) $(AM : M) = A + ann(M)$, for all ideals A of R.
(6) $(AM : BM) = (A + ann(M) : M)$, where A is a prime ideal of R and B is any ideal of R.

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