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Padovan Numbers by the Permanents of a Certain Complex Pentadiagonal Matrix

Diyar O.Mustafa Zangana[1] Ahmet Öteleş[2]

[1]Siirt University, Faculty of Science, Department of Mathematics, TR-56100, Siirt, Turkey
diyar.math2@gmail.com
[2]Dicle University, Education Faculty, Department of Mathematics, TR-21280, Diyarbakir/Turkey
aoteles85@gmail.com

Abstract
In this paper, we consider a certain type of complex pentadiagonal matrices. Then we show that the permanents of this matrix generate Padovan numbers. Finally, we give a Maple procedure in order to verify our result.

***Keywords*:** Permanent, Pentadiagonal matrix, Padovan number.

1. Introduction

The famous integer sequences (e.g., Fibonacci, Lucas, Padovan) provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory [1, 2]. Among these sequences, Padovan numbers have achieved a kind of celebrity status. The Padovan sequence $\{P(n)\}$ is defined by the recurrence relation, for $n > 2$

$$P(n) = P(n-2) + P(n-3)$$

with $P(0) = P(1) = P(2) = 1$[3]. The number $P(n)$ is called n^{th} Padovan number. The Padovan numbers are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, \ldots$$

for $n = 0, 1, 2, \ldots$ This sequence is named as A000931 in [4].
The permanent of a $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma}(i)$$

where the summation extends over all permutations σ of the symmetric group S_n. Thepermanent of a matrix is analogous to the determinant, where all of the signs used in theLaplace expansion of minors are positive.

Permanents have many applications in physics, chemistry, graph theory, electrical engineering, and so on [5, 6, 7, 8, 9]. One of the most important applications of permanents is the relationship between some special types of matrices and the well-known number sequences. There are many papers in relation to that applications. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] are some of them.

In this paper, we consider a certain type of complex pentadiagonal matrices. Then we show that the permanents of this matrix generate Padovan numbers. Finally, we give a Maple procedure in order to verify our result.

2. Main Results

Let $A = [a_{ij}]$be an $m \times n$ real matrix with row vectors $a_1, a_2, \ldots, a_m$. We say A iscontractible on column(resp. row) k if column (resp. row) kcontains exactly twonon-zero entries. Suppose A is contractibleon column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$matrix $A_{ij:k}$ obtained fromA by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A_{ij:k}^{T}\right]^{T}$ is called the contraction of A on row k relative to columns i and j. We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, \ldots, A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, \ldots, t$ [6].

Brualdi and Gibson [6] proved the following result about the permanent of a matrix.

Lemma 1 LetA be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A. Then

perA = perB. (1)

Let $H_n = h_{ij}$ be an $n \times n$ pentadiagonal matrix as the following

$$A = \begin{pmatrix} 1 & i & -1 & 0 & \cdots & \cdots & 0 \\ -i & 0 & i & -1 & 0 & & \vdots \\ 0 & -i & 0 & i & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & i & -1 \\ \vdots & & & 0 & -i & 0 & i \\ 0 & \cdots & \cdots & \cdots & 0 & -i & 0 \end{pmatrix}_{n\times n} . \qquad (2)$$

where $i = \sqrt{-1}$. If $n = 5$, then we obtain the permanent of H_5 by using Laplace
expansion as the following

$$\mathrm{Per}H_5 = \mathrm{Per}\begin{pmatrix} 1 & i & -1 & 0 & 0 \\ -i & 0 & i & -1 & 0 \\ 0 & -i & 0 & i & -1 \\ 0 & 0 & -i & 0 & i \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}_{5\times 5}$$

$$= \mathrm{Per}\begin{pmatrix} 0 & i & -1 & 0 \\ -i & 0 & i & -1 \\ 0 & -i & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} + (-1)\mathrm{Per}\begin{pmatrix} i & -1 & 0 & 0 \\ -i & 0 & i & -1 \\ 0 & -i & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

$$= (-i)\mathrm{Per}\begin{pmatrix} i & -1 & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix} + \mathrm{Per}\begin{pmatrix} 0 & i & -1 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix} - \mathrm{Per}\begin{pmatrix} -1 & 0 & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$= \mathrm{Per}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} - \mathrm{Per}\begin{pmatrix} -1 & 0 \\ -i & 0 \end{pmatrix} + (-i)\mathrm{Per}\begin{pmatrix} i & -1 \\ -i & 0 \end{pmatrix} + \mathrm{Per}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$= 1 - 0 + 1 + 1 = 3 = P(5).$$

By the contraction method introduced by Brualdi in [6], we now present the following theorem that gives the relationship between the permanent of the pentadiagonal matrix H_n and the Padovan number $P(n)$.

Theorem 2 Let H_n be the $n \times n$ pentadiagonal matrix given by (2). Then the permanent of the matrix is equal to the n^{th} Padovan number $P(n)$.

Proof. Let H_n^k be the k^{th} contraction of $H_n, 1 \leq k \leq n-2.$ Since the definition of the matrix H_n; thematrix $H_n$can be contracted on column 1 so that

$$H_n^1 = \begin{pmatrix} 1 & 2i & -1 & 0 & \cdots & \cdots & 0 \\ -i & 0 & i & -1 & 0 & & \vdots \\ 0 & -i & 0 & i & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & i & -1 \\ \vdots & & & 0 & -i & 0 & i \\ 0 & \cdots & \cdots & \cdots & 0 & -i & 0 \end{pmatrix}_{(n-1)\times(n-1)}$$

Since the matrix H_n^1 can be contracted on column 1

$$H_n^2 = \begin{pmatrix} 2 & 2i & -1 & 0 & \cdots & \cdots & 0 \\ -i & 0 & i & -1 & 0 & & \vdots \\ 0 & -i & 0 & i & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & i & -1 \\ \vdots & & & 0 & -i & 0 & i \\ 0 & \cdots & \cdots & \cdots & 0 & -i & 0 \end{pmatrix}_{(n-2)\times(n-2)}$$

Furthermore, the matrix H_n^2 can be contracted on column 1 and
$P(3) = P(4) = 2, P(5) = 3$ so that

$$H_n^3 = \begin{pmatrix} 2 & 3i & -2 & 0 & \cdots & \cdots & 0 \\ -i & 0 & i & -1 & 0 & & \vdots \\ 0 & -i & 0 & i & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & i & -1 \\ \vdots & & & 0 & -i & 0 & i \\ 0 & \cdots & \cdots & \cdots & 0 & -i & 0 \end{pmatrix}_{(n-3)\times(n-3)}$$

$$= \begin{pmatrix} P(4) & iP(5) & -P(3) & 0 & \cdots & \cdots & 0 \\ -i & 0 & i & -1 & 0 & & \vdots \\ 0 & -i & 0 & i & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & i & -1 \\ \vdots & & & 0 & -i & 0 & i \\ 0 & \cdots & \cdots & \cdots & 0 & -i & 0 \end{pmatrix}_{(n-3)\times(n-3)}$$

Continuing this process, we have

$$H_n^k = \begin{pmatrix} P(k+1) & iP(k+2) & -P(k) & 0 & \cdots & \cdots & 0 \\ -i & 0 & i & -1 & 0 & & \vdots \\ 0 & -i & 0 & i & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & i & -1 \\ \vdots & & & 0 & -i & 0 & i \\ 0 & \cdots & \cdots & \cdots & 0 & -i & 0 \end{pmatrix}_{(n-k)\times(n-k)}$$

for $3 \leq k \leq n-4$. Hence,

$$H_n^{n-3} = \begin{pmatrix} P(n-2) & iP(n-2) & -P(n-3) \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}_{3\times3}$$

which, by contraction of H_n^{n-3} on column 1, gives

$$H_n^{n-2} = \begin{pmatrix} P(n-1) & iP(n) \\ -i & 0 \end{pmatrix}_{2\times2}$$

By applying equation (1), we obtain $\mathrm{per}H_n = H_n^{n-2} = -i^2 P(n) = P(n)$ which is desired

2.1. Maple Procedure

The following Maple procedure calculates the permanent of the pentadiagonal matrix H_n given by (2).

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
```

```
local I,j,r,f,H;
f:=(i,j)->piecewise(i=1 and j=1,1,j-i=-1,-I,j-i=1,I,j-i=2,-1,0);
H:=Matrix(n,n,f):
for r from 0 to n-2 do
print(r,H):
for j from 2 to n-r do
H[1,j]:=H[2,1]*H[1,j]+H[1,1]*H[2,j]:
od:
H:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,H),1),2):
od:
print(r,eval(H)):
end proc:with(LinearAlgebra):
permanent(n);
```

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