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Some Properties of Preopen Set in Closure Spaces

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Abstract

Using the concept of preopen set, we introduce and study closure properties of pre-limit points, pre-derived sets, pre-interior and pre-closure of a set, pre-interior points, pre-border, pre-frontier and pre-exterior in closure space. The relations between pre-closure of a set and pre-interior (point) in closure spaces and pre-closure of a set and pre-interior (point) in topological space are investigated.

Keywords: Pre-limit point, Pre-derived set, Pre-closure, Pre-interior points, Pre-border of sets, Pre- frontier of sets, Pre-exterior points.

Introduction

The notion of (X, c)(closure space) was introduced by Khampakdee [1]. He introduced open set and closed set in closure space [1]. And also he introduced *Semi-open sets in biclosure spaces* [2]. The notion of preopen set was introduced by Mashhour et al [3]. In [4] Halgwrd M. Darwesh defined preopen set in closure space which is different of preopen set in topological space, he introduced and studied some properties of preopen sets in closure space. They work in operation in topology in [5-28]. In this paper, we introduce the notions of pre-limit points, pre-derived sets, pre-interior of sets. We study some results of topological spaces in [29] & [30].

2.Preliminaries

Through this paper, (X, τ) (resp. (X, c)) always mean topological spaces (closure spaces). The intersection of all closed sets in topological spaces which contain A, is called closure of set denoted by $Cl(A)$. And also the union of all open sets which contain in A is called interior of A which is denoted $Int(A)$. A subset A of X is said to be preopen [3] if $A \subseteq Int(Cl(A))$. The complement of a preopen set is called a preclosed set.

Definition 2.1 [1]

A function $c: P(X) \rightarrow P(X)$ defined on the power set $P(X)$ of a set X is called a closure operator on X and the pair (X, c) is called a closure space if the following axioms are satisfied:

$(A1)$ $c(\phi) = \phi$.
$(A2)$ $A \subseteq c(A)$ for every $A \subseteq X$.

$(A3)$ $A \subseteq B \Rightarrow c(A) \subseteq c(B)$ for all $A, B \subseteq X$. A closure operator c on a set X is called additive (respectively, idempotent) if $A, B \subseteq X$, $c(A \cup B) = c(A) \cup c(B)$ (respectively, for all $A \subseteq X \Rightarrow cc(A) = c(A)$. A subset $A \subseteq X$ is closed in the closure space (X, c) *if* $c(A) = A$. It is called open, if its complement in X is closed. The empty set and the whole space are both open and closed.

Definition 2.2 [4]

A subset A of a space (X, c) is said to be a preopen set, if there exists an open set G such that $A \subseteq G \subseteq c(A)$. The complement of a preopen set is called preclosed. The family of all preopen sets denoted by $PO(X, c)$. The family of all preclosed sets denoted by $PC(X, c)$.

Theorem 2.1 [4]

A subset A of a space (X, c) is preclosed if and only if there exists a closed set F such that $X \backslash c(X \backslash A) \subseteq F \subseteq A$.

Proposition 2.1 [4]

The union (intersection) of any family of preopen (preclosed) sets in a space(X, c) is preopen (preclosed).

Definition 2.3 [4]

The interior operator $i: P(X) \rightarrow P(X)$ corresponding to the closure operator c on X is given by; $i(A) = X \backslash c(X \backslash A)$.

Theorem 2.2 [4]

Let A be a subset of a closure (X, c). If $x \in c(A)$, then $G \cap A \neq \phi$, for each open subset G of X containing x.

Theorem 2.3 [4]

Let A be a subset of a closure (X, c) and c is idempotent on X, then $x \in c(A)$ if and only if $G \cap A \neq \phi$, for each open subset G of X containing x.

Proposition 2.2 [4]

Let c be an idempotent closure operator on a set X. If A is preopen in X and $B \subseteq A \subseteq c(B)$, then B is preopen.

Theorem 2.4 [4]

Let c be an idempotent closure operator on X. A subset A of X is preopen if and only if $A \subseteq X \backslash c(X \backslash cA) = ic(A)$.

Proposition 2.3 [4]

If A is closed and preopen in a space (X, c), then A is open.

3 Some Properties of Preopen Sets

Definition 3.1

Let (X, c) be a closure space, $x \in X$ and N be a subset of X. Then N is called a pre-neighborhood of x in X, if there exists a preopen set V_x such that $x \in V_x \subseteq N$.

Definition 3.2

Let A be a subset of a closure space $(X, c\ )$. A point $x \in X$ is said to be pre-limit point of A, if it satisfy the following assertion:

$V \cap (A \setminus \{x\}) \neq \phi$, for every preopen set V such that $x \in V$.The set of all pre-limit points of A is called the prederived set of A and is denoted by $D_P(A)$.

Note that for a subset A of X, a point $x \in X$ is not a pre-limit point of A if and only if there exists a preopen set V in X such that $x \in V$ such that $V \cap (A \setminus \{x\}) = \phi$, or (equivalently, $x \in V$ and $V \cap A = \phi$ or $V \cap A = \{x\}$).

Theorem 3.1

Let c_1 and c_2 be two closure operator on X such that $PO(X, c_1) \subseteq PO(X, c_2)$. For any subset A of X, every pre-limit point of A with respect to c_2 is a pre-limit point of A with respect to $c_{1.}$

Proof.

Let x be a pre-limit point of A with respect to c_2.Then $V \cap (A\ \ \{x\}) \neq \phi$, for every preopen set V with respect to c_2, such that $x \in V$. But $c_1 \subseteq c_2$, so, in particular, $V \cap (A\ \ \{x\}) \neq \phi$, for every preopen set V with respect to c_1, such that $x \in V$. Hence x is a pre-limit point of A with respect to c_1.

The converse of Theorem 3.1 is not true in general as seen in the following example.

Example 3.1

Let $X = \{a, b, c, d\}$ and defined closure operator $c_1 : P(X) \longrightarrow P(X)$ by:

$$c_1(A) = \begin{cases} \phi & \text{if } A = \phi \\ \{b, c, d\} & \text{if } \phi \neq A \subseteq \{b, c, d\} \\ X & otherwise \end{cases}$$

So $PO(X, c_1) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$.

And also defined closure operator $c_2 : P(X) \longrightarrow P(X)$ by:

$$c_2(A) = \begin{cases} \phi & if\ A = \phi \\ X & if\ \phi \neq A \subseteq X. \end{cases}$$

So $PO(X, c_2) = P(X)$. We have $PO(X, c_1) \subseteq PO(X, c_2)$. Let $A = \{a, b\}$, then $D_P(A) = \{b, c, d\}$ with respect $PO(X, c_1)$ and $D_P(A) = \phi$ with respect $PO(X, c_2)$. Note that d is pre-limit point of A with respect $PO(X, c_1)$, but it is not a pre-limit point of A with respect $PO(X, c_2)$.

Theorem 3.2

For any subsets A and B of (X, c), the following assertions are valid:

(1) If $A \subseteq B$, then $D_P(A) \subseteq D_P(B)$.

(2) $D_P(A) \cup D_P(B) \subseteq D_P(A \cup B)$.

(3) $D_P(A \cap B) \subseteq D_P(A) \cap D_P(B)$.

(4) $D_P(D_P(A)) \setminus A \subseteq D_P(A)$.

(5) $D_P(A \cup D_P(A)) \subseteq A \cup D_P(A)$.

Proof. (**1**): Let $x \in D_P(A)$. Then $x \in X$ is a pre-limit point of A. So $V \cap (A \setminus \{x\}) \neq \phi$, for every preopen set V. But $A \subseteq B$, so $V \cap (B \setminus \{x\}) \neq \phi$, then x is a pre-limit point of B, that is $x \in D_P(B)$. Hence, $D_P(A) \subseteq D_P(B)$.

(**2**) We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by(1) $D_P(A) \subseteq D_P(A \cup B)$ and $D_P(B) \subseteq D_P(A \cup B)$. Hence, $D_P(A) \cup D_P(B) \subseteq D_P(A \cup B)$.

(**3**)We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by(2) $D_P(A \cap B) \subseteq D_P(A)$ and $D_P(A \cap B) \subseteq D_P(B)$.Hence, $D_P(A \cap B) \subseteq D_P(A) \cap D_P(B)$.

(**4**) Let $x \in D_P\big(D_P(A)\big)\setminus A$. So $x \in D_P\big(D_P(A)\big)$ and $x \notin A$. Then, x is a pre-limit point of $D_P(A)$. That is, $V \cap (D_P(A) \setminus \{x\}) \neq \phi$, for every preopen set V. Then, there exists $y \in V \cap (D_P(A) \setminus \{x\})$. So $y \in V$ and $y \in D_P(A) \setminus \{x\}$. Then, $y \in D_P(A)$ and $y \neq x$. Thus, y is a pre-limit point of A. Then, $V \cap (A \setminus \{y\}) \neq \phi$ and $y \neq x$. If we take $z \in V \cap (A \setminus \{y\})$, so $x \neq z$ because $x \notin A$. Hence, $V \cap (A \setminus \{x\}) \neq \phi$, then x is a pre-limit point of A. Therefore $x \in D_P(A)$. Thus $D_P(D_P(A))\setminus A \subseteq D_P(A)$.

(**5**) Let $x \in D_P\big(A \cup D_P(A)\big)$. Then x is a pre-limit point of $A \cup D_P(A)$. If $x \in A$, the result is obvious. Assume that $x \notin A$. Then, $V \cap (A \cup D_P(A)\setminus \{x\}) \neq \phi$, for all preopen set V.This means that, $V \cap (A \setminus \{x\}) \neq \phi$ or $V \cap (D_P(A)\setminus \{x\}) \neq \phi$. The first case implies $x \in D_P(A)$. If $V \cap (D_P(A)\setminus \{x\}) \neq \phi$, then $x \in D_P\big(D_P(A)\big)$. Since $x \notin A$, it follows similarly from (4)that $x \in D_P(D_P(A))\setminus A \subseteq D_P(A)$. Therefore (5) is valid.

In general, neither inclusion of Theorem 3.2 is true as we will seen in the following examples.

Example 3.2

Let $X = \{a, b, c, d\}$ and defined closure operator $c: P(X) \longrightarrow P(X)$ by: $c(A) =$

$$\begin{cases} A & \text{if } A \in \{\phi, \{b\}, \{c\}, \{b, c\}\} \\ \{a, b\} & \text{if } A \in \{\{a\}, \{a, b\}\} \\ \{c, d\} & \text{if } A \in \{\{d\}, \{c, d\}\} \\ \{a, b, c\} & \text{if } A \in \{\{a, c\}, \{a, b, c\}\} \\ \{b, c, d\} & \text{if } A \in \{\{b, d\}, \{b, c, d\}\} \\ X & otherwise \end{cases}$$

Hence $PO(X, c) = \{\phi, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$. For two subsets $A = \{a, c\}$ and $B = \{a, b, d\}$ of X, we get $D_P(A) = \{b\} \subseteq \{b, c\} = D_P(B)$, but $A \nsubseteq B$. This shows that the converse of Theorem 3.2(1) is not valid.

Example 3.3

Let $X = \{a, b, c, d, e\}$ and defined closure operator $c: P(X) \longrightarrow P(X)$ by:

$$c(A) = \begin{cases} A & \text{if } A \in \{\phi\} = \mathcal{F} \\ \{b, e\} & \text{if } A \in \{\{b\}, \{e\}, \{b, e\}\} = \mathcal{G} \\ \{a, b, e\} & \text{if } A \in \{\{a\}, \{a, e\}, \{a, b\}, \{a, b, e\}\} = \mathcal{H} \\ \{b, c, d, e\} & \text{if } A \notin \{\mathcal{F}, \mathcal{G}, \mathcal{H}\} \text{and } A \subsetneq \{b, c, d, e\} \\ X & otherwise \end{cases}$$

$PO(X, c) = \{\phi, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, c, e\},$

$\{a, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, X\}$. Now consider two subsets $A = \{a, b\}$ and $B = \{b, c, d\}$ of X. Then $D_P(A) = \{b, e\}$, $D_P(B) = \{a, e\}$, and so $D_P(A \cap B) = \phi$, but $D_P(A) \cap D_P(B) = \{e\} \nsubseteq \phi = D_P(A \cap B)$. Thus the equality in Theorem 3.2 (5)is not valid.

Example 3.4

Let $X = \{a, b, c, d\}$ and defined closure operator $c: P(X) \longrightarrow P(X)$ by:

$$c(A) = \begin{cases} A & \text{if } A \in \{\phi, \{a\}, \{d\}, \{a, d\}\} \\ X & otherwise \end{cases}$$

Hence

$PO(X, c) =$
$\{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Let $A=\{a,b\}$ and $B=\{a,c\}$ be subsets of X. Then $D_P(A)=\phi=D_P(B)$. $D_P(A)\cup D_P(B)=\phi$, $D_P(A\cup B)=\{a,d\}$ but $D_P(A\cup B)=\{a,d\}\nsubseteq\phi=D_P(A)\cup D_P(B)$. Thus the equality in Theorem 3.2(2) **Error! Reference source not found.** is not valid. For a subset $A=\{a,b,c\}$of X, we have $D_P\big(D_P(A)\big)=D_P(\{a,d\})=\phi$. But $D_P(A)=\{a,d\}\nsubseteq D_P\big(D_P(A)\big)\backslash A=\phi$, and so the equality in Theorem 3.2(4) is not valid. Now for a subset $B=\{b,c\}$ of X, we get $D_P(B)=\{a,d\}$, and so $B\cup D_P(B)=X$ and $D_P(X)=\{a,d\}$, but $B\cup D_P(B)=X\nsubseteq D_P\big(B\cup D_P(B)\big)=\{a,d\}$. This shows that $D_P(A\cup D_P(A))=A\cup D_P(A)=X$. Hence the equality in Theorem 3.2(5) is not valid.

Definition 3.3

Let (X,c) be a closure space and $A\subseteq X$. The intersection of all preclosed sets containing A is called the pre-closure of A, denoted by $Cl_P(A)$.

Theorem 3.3

Let (X,c) be a closure space, $A,B\subseteq X$ then the following properties are true:

(1) $Cl_P(A)$ is preclosed set.

(2) $A\subseteq Cl_P(A)$.

(3) $Cl_P(A)$ is smallest preclosed set which containing A.

(4) If $A\subseteq B$, then $Cl_P(A)\subseteq Cl_P(B)$.

(5) $Cl_P(A)\cup Cl_P(B)\subseteq Cl_P(A\cup B)$.

(6) $Cl_P(A\cap B)\subseteq Cl_P(A)\cap Cl_P(B)$.

(7) A is preclosed set if and only if $A=Cl_P(A)$.

(8) $Cl_P\big(Cl_P(A)\big)=Cl_P(A)$.

Proof:

1. It follows from Definition 3.3 and Proposition 2.1.
2. Obvious.
3. From (1) and (2), we get $Cl_P(A)$ is preclosed set which containing A. It is enough to show $Cl_P(A)$ is smallest preclosed. Let L be any preclosed set with $A\subseteq L$. Then, L is one of the preclosed sets in which the intersection is taken it is mean $Cl_P(A)=\bigcap\{K, K$ is preclosed set and $A\subseteq K\}$. Hence $Cl_P(A)$ is the smallest preclosed set containing A.
4. BY (2) $\boldsymbol{B\subseteq Cl_P(B)}$, since$\boldsymbol{A\subseteq B}$, so $\boldsymbol{A\subseteq Cl_P(B)}$, but $\boldsymbol{Cl_P(A)}$ is the smallest preclosed set containing $\boldsymbol{A}$. So $\boldsymbol{Cl_P(A)\subseteq Cl_P(B)}$.

5. It follows from (4).

6. It follows from (4).

7. Let $A = Cl_P(A)$.Since $Cl_P(A)$ is preclosed set by (1), then A is a preclosed set. Conversely: Let A be a preclosed set. Then A is the smallest preclosed set which contains A. So $A = Cl_P(A)$ by (3).

8. Let $L = Cl_P(A)$. So L is a preclosed set by (1), then by (7) $L = Cl_P(L)$. Thus $Cl_P(A) = Cl_P\big(Cl_P(A)\big)$.

Theorem 3.4

Let (X, c) be a closure space and $A \subseteq X$. Then $x \in Cl_P(A)$ if and only if $A \cap V \neq \phi$, for all preopen set V which contains x.

Proof.

Let $x \in Cl_P(A)$ and suppose that $A \cap V = \phi$, for some preopen set V which contains x. This implies that X/V is a preclosed set and $A \subseteq X/V$. So $Cl_P(A) \subseteq Cl_P(X/V) = X/V$. This implies that $x \in X/V$, which is a contradiction. Therefore, $A \cap V \neq \phi$, for all preopen set V, Which contains x.

Conversely. If $x \notin Cl_P(A)$, then there exists a preclosed set K such that $A \subseteq K$ and $x \notin K$. Hence $X \setminus K$ is a preopen set which containing x and $A \cap (X \setminus K) \subseteq A \cap (X \setminus A) = \phi$. Which is a contradiction. Hence, $x \in Cl_P(A)$is valid.

Corollary 3.1

For any subset A of a closure space (X, c), we have $D_P(A) \subseteq Cl_P(A)$.

Proof.

Let $x \in D_P(A)$. Then, $A/\{x\} \cap V \neq \phi$, for all preopen set V which contains x. So $A \cap V \neq \phi$, for all preopen set V that contains x. Thus, by Theorem 3.4 $x \in Cl_P(A)$.

Theorem 3.5

For any subset A of a closure space (X, c), we have $Cl_P(A) = A \cup D_P(A)$.

Proof.

Let $x \in Cl_P(A)$. Assume that $x \notin A$ and let V be a preopen set with $x \in V$. Then, $A/\{x\} \cap V \neq \phi$, and so $x \in D_P(A)$. Hence $Cl_P(A) \subseteq A \cup D_P(A)$. The reverse inclusion is valid by $A \subseteq Cl_P(A)$ and Corollary 3.1.

Theorem 3.6

For a subset A of a closure space (X, c), we have A is preclosed if and only if $D_P(A) \subseteq A$.

Proof.

Assume that A is preclosed. Let $x \notin A$, i.e., $x \in X \setminus A$. Since $X \setminus A$ is preopen, so x is not a pre-limit point of A, i.e., $x \notin D_P(A)$, because $(X \setminus A) \cap (A \setminus \{x\}) = \phi$. Hence, $D_P(A) \subseteq A$. The reverse implication is followed by Theorem 3.5.

Corollary 3.2

Let A be a subset of a closure space (X, c). If F is a preclosed superset of A, then $D_P(A) \subseteq F$.

Proof.

By Theorem 3.2 (1) and Theorem 3.6, $A \subseteq F$ implies $D_P(A) \subseteq D_P(F) \subseteq F$.

Theorem 3.7

Let A and B be any subsets of a closure space (X, c) such that A is preopen. If the family of all preopen subsets of X is form a topology on X, then $A \cap Cl_P(B) \subseteq Cl_P(A \cap B)$.

Proof.

Let $x \in A \cap Cl_P(B)$. Then, $x \in A$ and $\in Cl_P(B) = B \cup D_P(B)$. If $x \in B$, then $x \in A \cap B \subseteq Cl_P(A \cap B)$. If $x \notin B$, then $x \in D_P(B)$ and so $B/\{x\} \cap V \neq \phi$, for all preopen set V containing x. Since A is preopen and $V \cap A$ is also a preopen set containing x. Hence, $V \cap (A \cap B) = (V \cap A) \cap B \neq \phi$, and consequently $x \in Cl_P(A \cap B)$. Therefore, $A \cap Cl_P(B) \subseteq Cl_P(A \cap B)$.

Example 3.5

Let $X = \{a, b, c, d\}$ and defined closure operator $c: P(X) \longrightarrow P(X)$ by:

$$c(A) = \begin{cases} A & \text{if } A \in \{\phi, \{a\}\} = \mathcal{F} \\ \{a, d\} & \text{if } A \in \{\{d\}, \{a, d\}\} = \mathcal{G} \\ \{a, c, d\} & \text{if } A \notin \{\mathcal{F}, \mathcal{G}\} \text{ and } A \subseteq \{a, c, d\} \\ X & otherwise \end{cases}.$$

Hence $PO(X, c) = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$

which is a topology on X. Consider the subset $A = \{a, b\}$ and $B = \{b, c\}$ of X, then $A \cap$

$Cl_P(B) = \{a, b\} \neq X = Cl_P(A \cap B)$. This shows that the equality in Theorem 3.7 is not true in general.

Example 3.6

The family of all preopen subsets of Example 3.4 does not form a topology on X and since for subsets $A = \{a, b\}$ and $B = \{b, c\}$ of the closure space X $A \cap Cl_P(B) = \{a, b\} \nsubseteq \{b\} = Cl_P(A \cap B)$. This shows that the conditions that the family of all preopen sets of X form a topology, in Theorem 3.6 is necessary and it can not be dropped.

Definition 3.4

A closure space(X, c) is said to be discrete if every subset of X is open set.

Note that

(1) An closure spaces (X, c) is discrete if and only if every subset of X is closed.
(2) If A is a subset of a discrete closure space (X, c), then $D_P(A) = \phi$.

Proposition 3.1

Let A be a subset of a closure space (X, c). If a point $x \in X$ is a pre-limit point of A, then x is also a pre-limit point of $A \backslash \{x\}$.

Proof. Obvious.

Definition 3.5

Let A be a subset of a closure space (X, c). A point $x \in X$ is called a pre-interior point of A, if there exists a preopen set V such that $x \in V \subseteq A$. The set of all pre-interior points of A is called the pre-interior of A and is denoted by $Int_P(A)$.

Proposition 3.2

For subsets A and B a closure space (X, c), the following assertions are valid.

(1) $Int_P(A)$ is the union of all preopen subsets of A.
(2) $Int_P(A)$ is the largest preopen set contained in A.
(3) A is preopen if and only if $A = Int_P(A)$.
(4) $Int_P(Int_P(A)) = Int_P(A)$.
(5) $Int_P(A) = A \backslash D_P(X \backslash A)$.
(6) $X \backslash Int_P(A) = Cl_P(X \backslash A)$.
(7) $X \backslash Cl_P(A) = Int_P(X \backslash A)$.
(8) If $A \subseteq B$, then $Int_P(A) \subseteq Int_P(B)$.
(9) $Int_P(A) \cup Int_P(B)) \subseteq Int_P(A \cup B)$.
(10) $Int_P(A \cap B) \subseteq Int_P(A) \cap Int_P(B)$.

Proof:

1. Let $\{V_i : i \in \Lambda\}$ be the collection of all preopen subsets of X contained in A. If $x \in Int_P(A)$, then, there exists $j \in \Lambda$ such that $x \in V_j \subseteq A$. Hence, $x \in \bigcup_{i\in\Lambda} V_i$, and so $Int_P(A) \subseteq \bigcup_{i\in\Lambda} V_i$. On the other hand, if $y \in \bigcup_{i\in\Lambda} V_i$, then $y \in V_k \subseteq A$ for some $k \in \Lambda$. Thus, $y \in Int_P(A)$, and so $\bigcup_{i\in\Lambda} V_i \subseteq Int_P(A)$. Accordingly, $Int_P(A) = \bigcup_{i\in\Lambda} V_i$.

2. Since $Int_P(A) = \bigcup_{G\subseteq A}\{G, G \ \textit{is preopen set}\}$, so by Proposition 2.1 $Int_P(A)$ is a preopen set. Also $Int_P(A) \subseteq A$. Now, to prove $Int_P(A)$ is the largest preopen set contained in A. Let H be any other preopen set that contained in A. Since H is a preopen set and $H \subseteq A$. So $H \subseteq \bigcup_{G\subseteq A}\{G, G \ \textit{is preopen set}\} = Int_P(A)$. That is $H \subseteq Int_P(A)$. Thus, $Int_P(A)$ is the largest preopen set contained in A.

3. Let A be a preopen set. Since $Int_P(A)$ is largest preopen contained in A and $A \subseteq A$, so $A \subseteq Int_P(A)$. And since $Int_P(A) \subseteq A$. Thus, $A = Int_P(A)$.Conversely: It follows from part(1).

4. Let $U = Int_P(A)$.So U is preopen set, then $U = Int_P(U)$ by (3). Thus, $Int_P(A) = Int_P\big(Int_P(A)\big)$.

5. If $x \in A\backslash D_P(X\backslash A)$, then $x \notin D_P(X\backslash A)$ and so there exists a pre-open set V containing x such that $V \cap (X\backslash A) = \phi$. Thus, $x \in V \subseteq A$ and hence $x \in Int_P(A)$. This shows that $A\backslash D_P(X\backslash A) \subseteq Int_P(A)$. Now let $x \in Int_P(A)$. Since $Int_P(A) \cap (X\backslash A) = \phi$, we have $x \notin D_P(X\backslash A)$. Therefore, $Int_P(A) = A\backslash D_P(X\backslash A)$.

6. Using (4) and Theorem 3.5, we have $X\backslash Int_P(A) = X\backslash(A\backslash D_P(X\backslash A)) = (X\backslash A) \cup D_P(X\backslash A) = Cl_P(X\backslash A)$.

7. Using (4) and Theorem 3.5, we have $Int_P(X\backslash A) = (X\backslash A)\backslash D_P(A) = X\backslash(A \cup D_P(A)) = X\backslash Cl_P(A)$.

8. Let $x \in Int_P(A)$, then x is pre-interior of A, so there exists a preopen set V such that $x \in V \subseteq A$, but $A \subseteq B$. So $x \in V \subseteq B$, then x is pre-interior of B. Hence, $x \in Int_P(B)$. Thus $Int_P(A) \subseteq Int_P(B)$.It follows from part (8).

9. It follows from part (8).

The converse of (8) in Proposition 3.2. is not true in general as seen in the following example:

Example 3.7

Let $X = \{a, b, c, d, e\}$ and defined closure operator $c: P(X) \longrightarrow P(X)$ by:

$$c(A)=\begin{cases} A & \text{if } A\in\{\phi,\{a\},\{b,c,d,e\}\}=\mathcal{F} \\ \{b,e\} & \text{if } A\in\{\{b\},\{e\},\{b,e\}\}=\mathcal{G} \\ \{a,b,e\} & \text{if } A\in\{\{a,b\},\{a,e\},\{a,b,e\}\}=\mathcal{H} \\ \{b,c,d,e\} & \text{if } A\notin\{\mathcal{F},\mathcal{G},\mathcal{H}\} \text{ and } A\subseteq\{b,c,d,e\} \\ X & otherwise \end{cases}$$

Let $A=\{a,b\}$ and $B=\{a,c,d\}$ be subsets of X. Then $Int_P(A)=\{a\}\subseteq Int_P(B)=\{a,c,d\}$.

Definition 3.6

For a subset A of a closure space (X,c), the set

(1) $B_P(A)=A\backslash Int_P(A)$ is called the pre-border of A.

(2) $Fr_P(A)=Cl_P(A)\backslash Int_P(A)$ is called the pre-frontier of A.

Remark 3.1

If A is a preclosed subset of X, then $B_P(A)=Fr_P(A)$.

Example 3.8

Let (X,c) be the closure space which is described in Example 3.7. Let $A=\{a,b,e\}$ be a subset of X. Then $Int_P(A)=\{a\}$, and so $B_P(A)=\{b,e\}$. Since $A=\{a,b,e\}$ is pre-closed, $Cl_P(A)=\{a,b,e\}$ and thus $Fr_P(A)=\{b,e\}$.

Example 3.9

Consider the closure space (X,c)which is given in Example 3.3. For a subset $A=\{b,c,d\}$ of X, we have $Int_P(A)=\{c,d\}$ and $Cl_P(A)=\{b,c,d,e\}$. Hence $B_P(A)=\{b\}$ and $Fr_P(A)=\{b,e\}$.

Proposition 3.3

For a subset A of a closure space (X,c), the following statements hold:

(1) $A=Int_P(A)\cup B_P(A)$.

(2) $Int_P(A)\cap B_P(A)=\phi$.

(3) A is a preopen set if and only if $B_P(A)=\phi$.

(4) $B_P(Int_P(A))=\phi$.

(5) $Int_P(B_P(A))=\phi$.

(6) $B_P\left(B_P(A)\right)=B_P(A)$.

(7) $B_P(A)=A\cap Cl_P(X\backslash A)$.

(8) $B_P(A) = A \cap D_P(X \setminus A)$.

Proof:

1. Obvious.

2. Obvious.

3. It follows from Proposition 3.2 (3) and Definition 3.5 (1).

4. Since $Int_P(A)$ is preopen, it follows from (3) that $B_P(Int_P(A)) = \phi$.

5. If $x \in Int_P(B_P(A))$, then $x \in B_P(A) \subseteq A$, and then $x \in Int_P(A)$. Thus, $x \in B_P(A) \cap Int_P(A) = \phi$, which is a contradiction. Hence, $Int_P(B_P(A)) = \phi$.

6. Using (5), we get $B_P(B_P(A)) = B_P(A) \setminus Int_P(B_P(A)) = B_P(A)$.

7. Using Proposition 3.2 (6) we have $B_P(A) = A \setminus Int_P(A) = A \setminus (X \setminus Cl_P(X \setminus A)) = A \cap Cl_P(X \setminus A)$.

8. Applying (7) and Theorem 3.5, we have we have to show $Cl_P(A) \subseteq A$. To this end, let $x \notin A$. Then $x \notin Fr_P(A)$. So $x \notin Cl_P(A) \setminus Int_P(A)$. But since $Int_P(A) \subseteq A$ and $x \notin A$, so $x \notin Cl_P(A)$. This means that, $Cl_P(A) \subseteq A$. So A is preclosed.

Lemma 3.1

For a subset A of a closure space (X, c), A is preclosed if and only if $Fr_P(A) \subseteq A$.

Proof.

Assume that A is preclosed. Then $Fr_P(A) = Cl_P(A) \setminus Int_P(A) = A \setminus Int_P(A) \subseteq A$. Conversely suppose that $Fr_P(A) \subseteq A$, then $Cl_P(A) \setminus Int_P(A) \subseteq A$.To show A is preclosed.. In view of Theorem 3.3(7). We have to show $Cl_P(A) \subseteq A$. To this end, let $x \notin A$. Then $x \notin Fr_P(A)$. So $x \notin Cl_P(A) \setminus Int_P(A)$. But since $Int_P(A) \subseteq A$ and $x \notin A$, so $x \notin Cl_P(A)$. This means that, $Cl_P(A) \subseteq A$. So A is preclosed.

Theorem 3.8

For a subset A of a closure space (X, c), the following assertions are valid:

(1) $Cl_P(A) = Int_P(A) \cup Fr_P(A)$.

(2) $Int_P(A) \cap Fr_P(A) = \emptyset$.

(3) $B_P(A) \subseteq Fr_P(A)$.

(4) $Fr_P(A) = B_P(A) \cup (D_P(A) \setminus Int_P(A))$.

(5) A is a preopen set if and only if $Fr_P(A) = B_P(X \setminus A)$.

(6) $Fr_P(A) = Cl_P(A) \cap Cl_P(X \setminus A)$.

(7) $Fr_P(A) = Fr_P(X \setminus A)$.

(8) $Fr_P(A)$ is preclosed.

(9) $Fr_P(Fr_P(A)) \subseteq Fr_P(A)$.
(10) $Fr_P(Int_P(A)) \subseteq Fr_P(A)$.
(11) $Fr_P(Cl_P(A)) \subseteq Fr_P(A)$.
(12) $Int_P(A) = A \setminus Fr_P(A)$.

Proof:

1. Obvious.
2. Obvious.
3. Obvious.
4. Using Theorem 3.5, we obtain $Fr_P(A) = Cl_P(A) \setminus Int_P(A) = (A \cup D_P(A)) \cap (X \setminus Int_P(A)) = (A \setminus Int_P(A)) \cup (D_P(A) \setminus Int_P(A)) = B_P(A) \cup (D_P(A) \setminus Int_P(A))$.
5. Assume that A is preopen. Then $Fr_P(A) = B_P(A) \cup (D_P(A) \setminus Int_P(A)) = \phi \cup (D_P(A) \setminus A) = D_P(A) \setminus A = B_P(X \setminus A)$ by using (4), Proposition 3.3 (3), Proposition 3.2 (3) and Proposition 3.3 (8).
   Conversely; suppose that $Fr_P(A) = B_P(X \setminus A)$. Then $\phi = Fr_P(A) \setminus B_P(X \setminus A) = \big(Cl_P(A) \setminus Int_P(A)\big) \setminus \big((X \setminus A) \setminus Int_P(X \setminus A)\big) = A \setminus Int_P(A) = B_P(A)$. By (5) and (6) of Proposition 3.2, and so by Proposition 3.3(3), A is preopen.
6. It follows from Proposition 3.2(6).
7. It is followed from (6).
8. we have $Cl_P(Fr_P(A)) = Cl_P(Cl_P(A) \cap Cl_P(X \setminus A)) \subseteq Cl_P(Cl_P(A)) \cap Cl_P(Cl_P(X \setminus A)) = Cl_P(A) \cap Cl_P(X \setminus A) = Fr_P(A)$. Obviously $Fr_P(A) \subseteq Cl_P(Fr_P(A))$, and so $Fr_P(A) = Cl_P(Fr_P(A))$. Hence $Fr_P(A)$ is preclosed.
9. This is by (8) and Lemma 3.1.
10. Proposition 3.2 (4), we get $Fr_P(Int_P(A)) = Cl_P(Int_P(A)) \setminus Int_P(Int_P(A)) \subseteq Cl_P(A) \setminus Int_P(A) = Fr_P(A)$.
11. We obtain $Fr_P(Cl_P(A)) = Cl_P(Cl_P(A)) \setminus Int_P(Cl_P(A)) \subseteq Cl_P(A) \setminus Int_P(A) = Fr_P(A)$.
12. We get $A \setminus Fr_P(A) = A \setminus (Cl_P(A) \setminus Int_P(A)) = A \cap ((X \setminus Cl_P(A)) \cup Int_P(A)) = \phi \cup (A \cup Int_P(A)) = Int_P(A)$.

The converse of (3) is not true in general as seen in the following example.

Example 3.10

Example 3.9 shows that the reverse inclusion of Theorem 3.9 (3) is not valid in general.

Definition 3.7

For a subset A of a closure space (X, c), the pre-interior of $X\backslash A$ is called the pre-exterior of A, and it is denoted by $Ext_P(A)$, that is, $Ext_P(A) = Int_P(X\backslash A)$.

Example 3.11

Consider the closure space (X, c)which is given in Example 3.7. For subsets $A = \{a, b, c\}$ and $B = \{b, d\}$ of X, we have $Ext_P(A) = \{d, e\}$ and $Ext_P(B) = \{a, c, e\}$.

Theorem 3.9

For subsets A and B of a closure space (X, c), the following assertions are valid.

(1) $Ext_P(A)$ is preopen.

(2) $Ext_P(A) = X\backslash Cl_P(A)$.

(3) $Int_P(A) \subseteq Int_P\big(Cl_P(A)\big) = Ext_P\big(Ext_P(A)\big)$.

(4) If $A \subseteq B$ then $Ext_P(B) \subseteq Ext_P(A)$.

(5) $Ext_P(A \cup B) \subseteq Ext_P(A) \cap Ext_P(B)$.

(6) $Ext_P(A) \cup Ext_P(B) \subseteq Ext_P(A \cap B)$.

(7) $Ext_P(X) = \phi, Ext_P(\phi) = X$.

(8) $Ext_P(A) = Ext_P(X\backslash Ext_P(A))$.

(9) $X = Int_P(A) \cup Ext_P(A) \cup Fr_P(A)$.

Proof.

1. It follows from Lemma 3.1 and Proposition 3.2(1).
2. It is straightforward by Proposition 3.2(7).
3. Applying (6) and (8) of Proposition 3.2, we get $Ext_P(Ext_P(A)) = Ext_P(Int_P(X\backslash A)) = Int_P(X\backslash Int_P(X\backslash A)) = Int_P(Cl_P(A)) \supset Int_P(A)$.
4. Assume that $A \subset B$, then $Ext_P(B) = Int_P(X\backslash B) \subseteq Int_P(X\backslash A) = Ext_P(A)$ by using Proposition 3.2(8).
5. Applying Proposition 3.2 (10), we get $Ext_P(A \cup B) = Int_P(X\backslash(A \cup B)) = Int_P((X\backslash A) \cap (X\backslash B)) \subseteq Int_P(X\backslash A) \cap Int_P(X\backslash B) = Ext_P(A) \cap Ext_P(B)$.
6. Using Proposition 3.2 (9), we obtain $Ext_P(A \cap B) = Int_P(X\backslash(A \cap B)) = Int_P((X\backslash A) \cup (X\backslash B) \supseteq Int_P(X\backslash A) \cup Int_P(X\backslash B) = Ext_P(A) \cup Ext_P(B)$.
7. $Ext_P(X) = Int_P(X\backslash X) = Int_P(\phi) = \phi$. Also $Ext_P(\phi) = Int_P(X\backslash \phi) = Int_P(X) = X$.
8. Using Proposition 3.2 (4), we have $Ext_P(X\backslash Ext_P(A)) = Ext_P(X\backslash Int_P(X\backslash A)) = Int_P(X\backslash A) = Ext_P(A)$.

9. $Int_P(A) \cup Ext_P(A) \cup Fr_P(A) = X\backslash Fr_P(A) \cup Fr_P(A) = X$.

Example 3.12

Let (X, c) be a closure space which is given in Example 3.7. Let $A = \{b, e\}$ and $B = \{c, d, e\}$. Then $Ext_P(B) = \{a\} \subseteq \{a, c, d\} = Ext_P(A)$. This shows that the converse of (4) in Theorem 3.9 is not valid. Now let $A = \{d, e\}$ and $B = \{c\}$. Then $Ext_P(A \cup B) = \{a\} \neq \{a, b\} = \{a, b, c\} \cap \{a, b, d, e\} = Ext_P(A) \cap Ext_P(B)$ which shows that the equality in Theorem 3.9 (5) is not valid. Finally let $A = \{a, b\}$ and $B = \{c, d, e\}$. Then $Ext_P(A \cap B) = \{a, b, c, d, e\}$ and $Ext_P(A) \cup Ext_P(B) = \{a, c, d, e\}$. This shows that the equality in Theorem 3.9 (6) is not valid.

Theorem 3.10

Let (X, c) be idempotent closure space and $A, B \subseteq X$, if $A \subseteq B$, then $i(A) \subseteq i(B)$.

Proof.

Since $A \subseteq B$, then $X\backslash B \subseteq X\backslash A$, so $c(X\backslash B) \subseteq c(X\backslash A)$ (since c is closure operator), then $X\backslash\, c(X\backslash A) \subseteq X\backslash\, c(X\backslash B)$, so $i(A) \subseteq i(B)$.

Theorem 3.11

Let (X, c) be idempotent closure space and $A \subseteq X$, then $i(i(A)) = i(A)$.

Proof. Let $i(A) = X\backslash c(X\backslash A)$, so $i(i(A)) = i\big(X\backslash c(X\backslash A)\big) = X\backslash c(X\backslash(X\backslash c(X\backslash A))) = X\backslash c(X\backslash(X\backslash c(X\backslash A))) = X\backslash c(c(X\backslash A))) = X\backslash c(c(X\backslash A))) = X\backslash c(c(X\backslash A))) = X\backslash c(X\backslash A) = i(A)$.

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