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On ω-Covering Dimension Functions

Alias B. Khalaf[1] and Halgwrd M. Darwesh[2]
[1]Mathematics Department, College of Science, University of Duhok, Kurdistan Region, Iraq
khalaf59@yahoo.com
[2]Mathematics Department, College of Science, University of Sulaimani, Kurdistan Region, Iraq.
halgwrd.darwesh@univsul.edu.iq

Abstract. The present paper is devoted to introduce and study a new type of covering dimension function of topological spaces by using ω-open sets. For this dimension function, some properties, characterizations and relationships with other concepts are found and proved.

1. Introduction and preliminaries.

The mathematician tried to know the dimension of spaces, before the definition of dimension was given; the use of dimension by mathematician was only vague sense, a space is n-dimensional if n is the least number of real parameters needed to describe its points in some unified way. In 19th century, there were two celebrate discovering, the first one, is the Cantor's one-to-one correspondence between a line and plane which it was shown that the one-to-one correspondence mapping cannot preserve the dimension. However, the second one, is the Peano's continuous mapping of the unit interval onto unit square, this shows that the definition of dimensions via parameters is not suitable. So, the mathematicians hoped the dimension has a topological meaning, till it is topological invariant. For the first time, the covering dimension function is made by Cech in 1933 and it is also studied by Lebesque.

Throughout this work, a space will always mean a topological space, $(X;\tau)$ and $(Y;\sigma)$ (or simply, X and Y) will denote spaces on which no separation axioms are assumed unless explicitly stated. The notations T_{dis}, and T_{ind} denote the discrete and indiscrete topologies and $\mathcal{U}$ denotes the usual topology for the set of all real numbers R. A point x in a space X is called a condensation point of $A \subseteq X$ [11, page 90], if $G \cap A$ is an uncountable set, for each open set G which contains x. A is said to be an ω–closed set [2], if it contains all its condensation points. The complement of an ω–closed set is called ω–open, and it is well known that, a subset A of a space X is ω–open if and only if for each $x \in A$, there exists an open set G contains x such that $G - A$ is a countable set [12]. The family of all ω-open sets of a space (X,τ) form a finer topology than τ and it is denoted by τ^{ω}, For a space (X,τ) we shall denote the space (X,τ^{ω}) by X^{ω}, and for any subset A of X, we denote by ClA, $Cl_{\omega}A$, $IntA$ and $Int_{\omega}A$, the closure, ω-closure, interior and ω-

interior of A in X. We recall the following definition and result which are needed to prove our results.

Definition 1.1. [12] A space X is said to be:

1. Locally countable, if each point of X contained in a countable open set,
2. Anti-locally countable, if each nonempty open subset of X is uncountable.

Definition 1.2. [1, p. 54] Let X be a space, the order of a family $\{B_\lambda; \lambda \in \Lambda\}$ of the subsets of X, not all empty, is the largest integer n, for which there exists a subset Δ of Λ with $n+1$ elements such that $\bigcap_{\lambda\in\Delta} B_\lambda$ is nonempty, or ∞ if there is no such largest integer. A family of empty subsets has order -1.

Theorem 1.3. [8, p. 24] Let $\{U_\lambda; \lambda \in \Lambda\}$ be a locally-finite family of open subsets of a normal space X and let $\{F_\lambda; \lambda \in \Lambda\}$ be a family of closed sets such that $F_\lambda \subseteq U_\lambda$ for each $\lambda \in \Lambda$. Then, there exists a family $\{G_\lambda; \lambda \in \Lambda\}$ of open sets such that $F_\lambda \subseteq G_\lambda \subseteq ClG_\lambda \subseteq U_\lambda$ for each $\lambda \in \Lambda$, and the families $\{F_\lambda; \lambda \in \Lambda\}$ and $\{ClG_\lambda; \lambda \in \Lambda\}$ are similar.

Theorem 1.4. [12] For any space (X,τ) and any subset A of X, we have:

1. $\tau^{\omega\omega} = (\tau^\omega)^\omega = \tau^\omega$ $(i.e., X^{\omega\omega} = (X^\omega)^\omega = X^\omega)$.
2. $(\tau_A)^\omega = (\tau^\omega)_A = \tau_A^\omega = A^\omega$.

Lemma 1.5. [12] For an anti-locally countable space X, we have:

1. $Cl_\omega A = ClA$, for each ω-open subset A of X.
2. $Int_\omega A = IntA$, for each ω-closed subset A of X.

Definition 1.6. [7] A space (X,τ) is called an ω-connected space provided that X is not the union of two nonempty disjoint ω-open sets. Analogously, (X,τ) is ω-disconnected, if it is not ω-connected.

Definition 1.7. [7] A space (X,τ) is called an ω–space if $\tau^\omega = \tau$ (i.e., $X^\omega = X$).

Theorem 1.8. [6] A space X is an ω-normal space if for each pair of ω-open sets U and V in X such that $X = U \cup V$, there exist ω-closed sets A and B which are contained in U and V, respectively and $X = A \cup B$.

Theorem 1.9. [6] Let X be an anti-locally countable space. If X is ω-normal (resp., ω-regular), then it is normal (resp. regular) and ω-space.

For any non-defined concepts see our references.

2. The ω-Covering Dimension Function Properties and Relationships

In this section, like the definition of covering dimension, we define another covering dimension which we call the $\omega-$covering dimension, and study some of its properties and relationships with other concepts.

Definition 2.1. The $\omega-$covering dimension of a space X is denoted by $\omega-\dim X$ and it is defined as follows:

$\omega-\dim X=-1$ if and only if X is empty. We say $\omega-\dim X\leq n$, where n is a non-negative integer, if each finite $\omega-$open covering of X has an $\omega-$open refinement of order not exceeding n. Also we say $\omega-\dim X=n$, if it is true that $\omega-\dim X\leq n$ but not $\omega-\dim X\leq n-1$. Finally, we say $\omega-\dim X=\infty$ if for any integer n, there exists a finite $\omega-$open covering X which has no $\omega-$open refinement of order not exceeding n.

Remark 2.2. Let Y be any subset of a space X. Then, we say $\omega-dinY\leq n$ if it is true as a subspace.

The following result shows that the $\omega-$covering dimension is monotonic on $\omega-$closed subspaces:

Proposition 2.3. If F is an $\omega-$ closed subspace of a space X, then $\omega-\dim F\leq\omega-\dim X$.

Proof. If $\omega-\dim X=\infty$ or $\omega-\dim X=-1$, then there is nothing to prove. So it is sufficient when we show that if $\omega-\dim X=n$, then $\omega-\dim F\leq n$. For this, let, $\{U_i\}_{i=1}^{t}$ be a finite covering of F by $\omega-$open sets of F. Then, by part (2) of Theorem 1.4, there exist $\omega-$ open sets V_i in X such that $U_i=V_i\cap F$ for each $i=1,2,\ldots,t$. Hence, $\{V_i\}_{i=1}^{t}\cup\{X-F\}$ is a finite $\omega-$ open covering of X. Since $\omega-\dim X=n$, then there exists an $\omega-$ open refinement $\{G_\lambda\}_{\lambda\in\Lambda}$ of $\{V_i\}_{i=1}^{t}\cup\{X-F\}$ of order not exceeding n. Thus, $\{G_\lambda\cap F\}_{\lambda\in\Lambda}$ is an $\omega-$ open refinement of $\{U_i\}_{i=1}^{t}$ of order not exceeding n. This implies that $\omega-\dim F\leq n$.

It is easy to show the following relationship between $\omega-$covering dimension and locally-countable spaces:

Proposition 2.4. If X is any nonempty locally-countable space, then $\omega-\dim X=0$.

Proof. Obvious.

The following example shows that the converse of Proposition 2.4 is not true:

Example 2.5. Consider the subspace $(Irr,\mathcal{U}_{Irr})$ of the usual space $(R,\mathcal{U})$, since $(Irr,\mathcal{U}_{Irr})$ is an anti-locally countable $\omega-$normal space and $\dim Irr=0$. Then by Theorem 1.9, we have $\omega-\dim Irr=0$.

The following proposition gives the relationship between $\omega-$covering dimension and $\omega-$normal spaces:

Proposition 2.6. If X is any space with $\omega-\dim X=0$, then X is $\omega-$normal.

Proof. Let $\omega-\dim X=0$ and U, V be two $\omega-$ open sets of X such that $U\cup V=X$. Therefore, there exists an $\omega-$open refinement $\{G_\lambda\}_{\lambda\in\Lambda}$ of the cover $\{U,V\}$ of order not exceeding 0. This means that the members of $\{G_\lambda\}_{\lambda\in\Lambda}$ are pairwise disjoint. Then $G = \bigcup_{\lambda\in\Lambda}\{G_\lambda ; G_\lambda \subseteq U\}$ and $W = \bigcup_{\lambda\in\Lambda}\{G_\lambda ; G_\lambda \subseteq V\}$ are disjoint and $X = G \bigcup W$. Therefore, by Theorem 1.8, X is an $\omega-$normal space.

The following example shows that the converse of Proposition 2.6 is not true in general.

Example 2.7. Consider the closed ordinal space $X = [0,\Omega]$ that is given in [6, Example 3.4]. Since X is a normal $\omega-$space, then X is an $\omega-$normal space and $\omega-\dim X = \dim X$. But since $\{\Omega\}$ is a closed subset of X and there is no clopen subset which contains $\{\Omega\}$. Since X is T_1 space, then $\dim X \neq 0$, hence $\omega-\dim X \neq 0$.

The following proposition gives the relationship between $\omega-$ covering dimension and $\omega-$disconnected spaces:

Proposition 2.8. Let X be any space with more than one point. If $\omega-\dim X = 0$, then X is $\omega-$disconnected.

Proof. Let $\omega-\dim X = 0$, and let x and y be two distinct points in X. Then, $\{X-\{x\}, X-\{y\}\}$ is a finite $\omega-$open covering of X. So by putting $U = X-\{x\}$ and $V = X-\{y\}$ in the proof of Proposition 4.1.6. Then, we obtain two disjoint $\omega-$ clopen sets $G \subseteq U$ and $W \subseteq V$ such that $G \bigcup W = X$. Thus G is a proper $\omega-$clopen subset of X. Hence, the space X is $\omega-$disconnected.

The following example shows that the converse of Proposition 2.8 is not true in general.

Example 2.9. Consider that the space $(R,\Im)$ with $\Im = \{\emptyset, \{0\}, R\}$. Since $\{0\}$ is an $\omega-$clopen subset of $(R,\Im)$, then $(R,\Im)$ X is $\omega-$disconnected. Also since there is no disjoint $\omega-$ open subset of $(R,\Im)$, except $\{0\}$ and $R-\{0\}$. This implies that $(R,\Im)$ is not an $\omega-T_2$ space. Hence by [6, Corollary 4.5] and Theorem 1.9, it is not $\omega-$normal. So by Proposition 2.6, $\omega-\dim R > 0$

The following examples show that the $\omega-$covering dimension ($\omega-\dim$) and covering dimension ($\dim$) are distinct. They also show that ($\omega-\dim$) is distinct from each of c- $\dim$ (s- $\dim$, s- $\dim_c$, p- $\dim$ and q- $\dim$). For these inductive dimensions we refer [3], [4], [5], [9] and [10]:

Example 2.10. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$. Then, $SO(X) = \tau = CO(X) = PO(X)$. Since the family $\{\{a\},\{a,b\},\{a,c\}\}$ is an open (semi-open, c-open, p-open) refinement of every open, clopen, semi-open, preopen cover of X, then $\dim X = 1 =$ c- $\dim X =$ s- $\dim X =$ s- $\dim_c X =$ p- $\dim X$, but by Proposition 2.4, we have $\omega-\dim X = 0$.

Example 2.11. Consider the topological space (X, T_{ind}), where X is an uncountable set. So we have $SO(X) = T_{ind} = CO(X) = (T_{ind})_q$ and $PO(X) = T_{dis}$. Then $\dim X = 0 =$ c-

$\dim X =$ s- $\dim X =$ s- $\dim_C X =$ p- $\dim X =$ q- $\dim X$. Since X is not $\omega-$ regular then $\omega-\dim X > 0$ (in fact $\omega-\dim X = \infty$).

However, the following results exhibit a relationship between $\omega-$covering dimension and covering dimension:

Corollary 2.12. If a space X is locally-countable, then $\omega-\dim X \le \dim X$.

Proof. Follows from Proposition 2.4.

Corollary 2.13. If a space X is anti-locally-countable $\omega-$regular or $\omega-$normal, then $\omega-\dim X = \dim X$.

Proof. Follows from Theorem 1.9.

Also, we obtain the following corollary:

Corollary 2.14. If a space X is anti-locally-countable such that $\omega-\dim X = 0$, then $\dim X = 0$.

Proof. Follows From Corollary 2.13.

Since if X is a countable set equipped with the discrete topology or indiscrete, then $\omega-\dim X = 0 = \dim X$. However, X is not an anti-locally-countable space. This means that the converse of the Corollary 2.13 is not true, and in virtue of Example 2.11, the $\omega-$regularity of X cannot be dropped in the Corollary 2.13, but it can be replaced by another condition for example see Corollary 2.17, below.

We can show the following relationship between $\omega-$covering dimension and covering dimension:

Theorem 2.15. Let X be an anti-locally-countable normal space. Then, $\dim X \le \omega-\dim X$.

Proof. If either $\omega-\dim X = \infty$ or $\omega-\dim X = -1$, then there is nothing to prove. Let n be any non-negative integer such that $\omega-\dim X = n$, and let $\{G_i\}_{i=1}^{t}$ be any finite open covering of X . Since X is normal, so, there exists an open covering $\{O_i\}_{i=1}^{t}$ such that $ClO_i \subseteq G_i$ for each $i = 1,2,...,t$. Again, by normality of X and Theorem 1.3, there exists a family $\{V_i\}_{i=1}^{t}$ of open subsets of X such that $ClO_i \subseteq V_i \subseteq ClV_i \subseteq G_i$ for each $i = 1,2,...,t$, and the families $\{ClV_i\}_{i=1}^{t}$ and $\{ClO_i\}_{i=1}^{t}$ are similar. Since $\omega-\dim X = n$, then there is an $\omega-$ open refinement ϑ of $\{V_i\}_{i=1}^{t}$ of order not exceeding n . Let $U_i = \bigcup\{W \in \vartheta;\ W \subseteq V_i\}$ for each $i = 1,2,...,t$. Clearly $\{U_i\}_{i=1}^{t}$ is of order not exceeding n. Since X is anti-locally-countable, and for each $i = 1,2,...,t$, U_i is $\omega-$open in X . Then by Lemma 1.5, we have $U_i \subseteq IntClU_i \subseteq ClV_i \subseteq G_i$. Hence, $\{IntClU_i\}_{i=1}^{t}$ is an open refinement of $\{G_i\}_{i=1}^{t}$ of order not exceeding n . Therefore, $\dim X \le n = \omega-\dim X$.

Finally, we have the following relationship between $\omega-$covering dimension and covering dimension:

Theorem 2.16 If a space X has the property that every $\omega-$open covering of X has a locally-finite open refinement. Then, $\omega-\dim X \le \dim X$.

Proof. Let X be a space with the given property. So, if either $\omega-\dim X = \infty$ or $\omega-\dim X = -1$, then there is nothing to prove. Suppose that $\dim X = n$, and let $\{U_i\}_{i=1}^{t}$ be any finite $\omega-$open covering of X. Then by hypothesis, this $\omega-$open covering has a locally-finite open refinement $\{G_\lambda\}_{\lambda\in\Lambda}$. Let $G_i = \bigcup_{\lambda\in\Lambda}\{G_\lambda ;\, G_\lambda \subseteq U_i\}$ for each i. Thus $\{G_i\}_{i=1}^{t}$ is a finite open covering of X. Since $\dim X = n$, then there exists an open refinement $\{W_\gamma\}_{\gamma\in\Gamma}$ of $\{G_i\}_{i=1}^{t}$ (and hence of $\{U_i\}_{i=1}^{t}$) of order not exceeding n, therefore, $\omega-\dim X \le n = \dim X$.

As an immediate consequence of Theorem 2.15 and Theorem 2.16, we have:

Corollary 2.17. If X an anti-locally-countable normal space with the property that every $\omega-$open covering of X has a locally-finite open refinement, then $\omega-\dim X = \dim X$.

3. Some Characterizations and Other Results on $\omega-$Covering Dimension

In this section, we give some characterizations of $\omega-$covering dimension. Also we give some other results on $\omega-$covering dimension, without proof and then our first characterization is the following:

Theorem 3.1. If X is any space, then, the following statements are equivalent:

1. $\omega-\dim X \le n$.
2. For every finite $\omega-$open covering $\{U_i\}_{i=1}^{t}$ of X, there is an $\omega-$open covering $\{V_i\}_{i=1}^{t}$ of order not exceeding n such that $V_i \subseteq U_i$ for each $i = 1,2,...,t$.
3. If $\{U_i\}_{i=1}^{n+2}$ is an $\omega-$open covering of X, there is an $\omega-$open covering $\{V_i\}_{i=1}^{n+2}$ such that $\bigcap_{i=1}^{n+2} V_i = \varnothing$.

Theorem 3.2. If X is any $\omega-$normal space, then, the following statements are equivalent:

1. $\omega-\dim X \le n$.
2. For every finite $\omega-$open covering $\{U_i\}_{i=1}^{t}$ of X there is an $\omega-$open covering $\{V_i\}_{i=1}^{t}$ such that $\omega ClV_i \subseteq U_i$ for each $i = 1,2,...,t$, and the order of $\{\omega ClV_i\}_{i=1}^{t}$ does not exceed n.

3. For every finite $\omega-$ open covering $\{U_i\}_{i=1}^{t}$ of X there is an $\omega-$ closed covering $\{F_i\}_{i=1}^{t}$ of order not exceeding n such that $F_i \subseteq U_i$ for each $i = 1,2,...,t$.
4. Every finite $\omega-$open covering of X has a finite $\omega-$closed refinement of order that does not exceed n.
5. If $\{U_i\}_{i=1}^{n+2}$ is an $\omega-$ open covering of X there is an $\omega-$ closed covering $\{F_i\}_{i=1}^{n+2}$ such that $\bigcap_{i=1}^{n+2} F_i = \varnothing$.

Theorem 3.3. If X is any $\omega-$normal space, then the following statements are equivalent:

1. $\omega-\dim X \le n$.
2. For each family $\{F_i\}_{i=1}^{n+1}$ of $\omega-$closed sets and each family $\{U_i\}_{i=1}^{n+1}$ of $\omega-$open sets of X such that $F_i \subseteq U_i$ for each i, there is a family $\{V_i\}_{i=1}^{n+1}$ of $\omega-$open sets such that $F_i \subseteq V_i \subseteq \omega ClV_i \subseteq U_i$ for each i, and $\bigcap_{i=1}^{n+1} \omega b(V_i) = \varnothing$.
3. For each family $\{F_i\}_{i=1}^{t}$ of $\omega-$closed sets and each family $\{U_i\}_{i=1}^{t}$ of $\omega-$open sets of X such that $F_i \subseteq U_i$ for each i, there exist families $\{V_i\}_{i=1}^{t}$ and $\{W_i\}_{i=1}^{t}$ of $\omega-$open sets such that $F_i \subseteq V_i \subseteq \omega ClV_i \subseteq W_i \subseteq U_i$ for each i, and the order of $\{\omega Cl(W_i) - V_i\}_{i=1}^{t}$ does not exceed $n-1$.
4. For each family $\{F_i\}_{i=1}^{t}$ of $\omega-$closed sets and each family $\{U_i\}_{i=1}^{t}$ of $\omega-$open sets of X such that $F_i \subseteq U_i$ for each i, there is a family $\{V_i\}_{i=1}^{t}$ of $\omega-$open sets such that $F_i \subseteq V_i \subseteq \omega ClV_i \subseteq U_i$ for each i, and the order of $\{\omega b(V_i)\}_{i=1}^{t}$ does not exceed $n-1$.

Similarly, as [26, p. 118], we can extend Theorem 3.3 to countable families in the following result:

Proposition 3.4. If X is any $\omega-$normal space such that $\omega-\dim X \le n$, then for each family $\{F_i\}_{i\in N}$ of $\omega-$closed sets and each family $\{U_i\}_{i\in N}$ of $\omega-$open sets of X such that $F_i \subseteq U_i$ for each i, there is a family $\{V_i\}_{i\in N}$ of $\omega-$open sets such that $F_i \subseteq V_i \subseteq U_i$ for each i, and the family $\{\omega b(V_i)\}_{i\in N}$ does not exceed $n-1$.

Definition 3.5. Let A and B be any two disjoint sets in a space X. A subset L is called an $\omega-$partition between A and B, if there exist two disjoint $\omega-$open sets U and W such that $A \subseteq U$, $B \subseteq W$ and $X - L = U \cup W$.

We can prove the following useful characterization of $\omega-$ covering dimension:

Theorem 3.6. If X is any $\omega-$normal space, then the following statements are equivalent:

1. $\omega-\dim X \le n$.
2. For each family $\{(E_i, F_i)\}_{i=1}^{n+1}$ of $n+1$ pairs of disjoint $\omega-$closed sets, there exist $n+1$ $\omega-$continuous mappings $f_i : X \to I$ such that $f_i(E_i)=\{0\}$ and $f_i(F_i)=\{1\}$ for each i, and $\bigcap_{i=1}^{n+1} f_i^{-1}\left(\frac{1}{2}\right)=\varnothing$.
3. For each family $\{(E_i, F_i)\}_{i=1}^{n+1}$ of $n+1$ pairs of disjoint $\omega-$closed sets, there exists a family $\{L_i\}_{i=1}^{n+1}$ of $\omega-$ closed sets of X such that L_i is an $\omega-$partition between E_i and F_i for each i, and $\bigcap_{i=1}^{n+1} L_i = \varnothing$.

The following result is called the sum theorem for $\omega-$ covering dimension:

Theorem 3.7. Let X be a topological sum of the family of spaces $\{X_\lambda\}_{\lambda\in\Lambda}$. If $\omega-\dim X_\lambda \le n$, for each $\lambda \in \Lambda$ then $\omega-\dim X \le n+1$.

The following is a useful theorem about $\omega-$covering dimension:

Theorem 3.8. If X is an $\omega-$normal space with the property that for each $\omega-$closed set F and each $\omega-$open set U such that $F \subseteq U$, there exists an $\omega-$ open set V in X such that $F \subseteq V \subseteq U$ and $\omega-\dim \omega b(V) \le n$. Then $\omega-\dim X \le n+1$.

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