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Trigonometric B-Spline Interpolation

Najim Abdullah Ismaeel

University of Garmian -College of Education - Math. Department

Abstract:

In present paper

the objective of the choice for studding trigonometric B-spline is made to show it is gives better approximate result or not of the boundary value problems in ordinary differential equations. By applying B-spline procedures to obtain approximate solution of the boundary value problems of ordinary differential equations with trigonometric B-spline, cubic trigonometric B-spline have motivated the solve of boundary value problems with numerical procedures.

Keywords:

Trigonometric B-spline, singular perturbed, second order boundary value problem.

2.Trigonometric B-Splines: [3]

Let $\{x_i\}$ be a non-decreasing sequence of real numbers such that $x_{i+k}-x_i<2\pi$ for all i, where k≥1 is a given integer. The real-valued functions $T_{i,k}$ on R defined by $T_{i,k}(x)=0$ if $x_{i+k}=x_i$ and $T_{i,k}(x)=[x_i, x_{i+1},\dots,x_{i+k}]_t\,(\sin\frac{y-x}{2})^{k-1}$ if $x_{i+k}>x_i$..

Definition 2.1 [28]:

The normalized trigonometric B-splines $T_{i,k}$ associated with the knot sequence $\{x_i\}$ which gives higher degree trigonometric B-splines, gives by the following iterative formula

$$T_{i,k}(x)=\frac{\sin\left(\frac{x-x_i}{2}\right)}{\sin\left(\frac{x_{i+k-1}-x_i}{2}\right)}T_{i,k-1}(x)+\frac{\sin\left(\frac{x_{i+k}-x}{2}\right)}{\sin\left(\frac{x_{i+k}-x_{i+1}}{2}\right)}T_{i+1,k-1}(x),k=2,3,4, \qquad (1)$$

starting with uniform normalized trigonometric B-spline

$$T_{i,1}(x)=\begin{cases}1 & for\ x_i \le x < x_{i+1}\\ 0 & otherwise\end{cases}$$

the $T_{i,k}$ functions as defined in (1) has the following properties :

i. Support $(T_{i,k})=[x_i , x_{i+1})$

ii. $T_{i,k} \geq 0$ for all x and all i,(is positive in the interior of its support and zero otherwise).

iii. $\sum_{i=-\infty}^{\infty} T_{i,k}$ (x)=1for all $x \in R$.

Trigonometric B-splines $T_{i,k}$ obtained by applying a linear factor to $T_{i,k-1}$ and $T_{i+1,k-1}$,we see that degree actually increased by 1 at each step .

The spline function S(x) with respect to the given trigonometric B-spline defined by

$$S(x)=\sum_{i=1}^{n} c_i T_i^m(x), c_i \in \mathbb{R}, i = 1, 2, \dots, n. \qquad (2)$$

2.1 Cubic Trigonometric B-spline: [4, 5]:

Let π be a uniform partition of the problem domain [a, b] such that $\pi =\{ a=x_0< x_1 < x_2 < \dots < x_{n-1} <x_n=b\}$, at the knot points x_i , i=0,…,n-1, $x_i=x_0+ih$ and mesh distance h=(b-a)/n, on this partition. The Cubic Trigonometric B-spline defined upon the set of n+1 knot points of the problem domain [a, b] as:

$$TB_{j,3}(x)=\frac{1}{\emptyset}\begin{cases}\sin^3\left(\frac{x-x_{i-2}}{2}\right), & [x_{i-2}, x_{i-1}]\\ \sin^2\left(\frac{x-x_{i-2}}{2}\right)\sin\left(\frac{x_i-x}{2}\right)+ & \\ \sin\left(\frac{x-x_{i-2}}{2}\right)\sin\left(\frac{x_{i+1}-x}{2}\right)\sin\left(\frac{x-x_{i-1}}{2}\right) & \\ +\sin^2\left(\frac{x-x_{i-1}}{2}\right)\sin\left(\frac{x_{i+2}-x}{2}\right), & [x_{i-1}, x_i]\\ \sin\left(\frac{x-x_{i-2}}{2}\right)\sin^2\left(\frac{x_{i+1}-x}{2}\right)+ & \\ \sin\left(\frac{x-x_{i-1}}{2}\right)\sin\left(\frac{x_{i+1}-x}{2}\right)\sin\left(\frac{x_{i+2}-x}{2}\right) & \\ +\sin\left(\frac{x-x_i}{2}\right)\sin^2\left(\frac{x_{i+2}-x}{2}\right), & [x_i, x_{i+1}]\\ \sin^3\left(\frac{x_{i+2}-x}{2}\right) & [x_{i+1}, x_{i+2}]\\ 0, & otherwise\end{cases}$$

where $\emptyset=\sin(h)\sin(\frac{h}{2})\sin(\frac{3h}{2})$

It is worth mentioning that $CTB_j\,(x)$ are twice continuously differentiable piecewise on the problem domain [a, b]. Now let S(x) be the spline interpolating function at the nodal points, then S(x) can be written as $S(x)=\sum_{j=-1}^{n+1} c_j TB_{j,3}$.

S(x) is approximate solution of differential equation where C_j's are unknown coefficients, and $TB_{j,3}(x)$ are cubic trigonometric B-spline functions. To solve boundary value problem of second order with using cubic trigonometric B-spline functions CTB_j. It required CTB_j, CTB_j', and CTB_j'' been evaluated at the nodal points, that are summarized in the following table 1.

Table 1. The values $B_{i,3}$, $B'_{i,3}$ and $B''_{i,3}$

X	X_{i-2}	X_{i-1}	X_i	X_{i+1}	X_{i+2}
CTB_j	0	$\frac{1}{2}\tan(\frac{h}{2})\csc(\frac{3h}{2})$	$2\sin(\frac{h}{2})\csc(\frac{3h}{2})$	$\frac{1}{2}\tan(\frac{h}{2})\csc(\frac{3h}{2})$	0
CTB'_j	0	$\frac{3}{4}\csc(\frac{3h}{2})$	0	$-\frac{3}{4}\csc(\frac{3h}{2})$	0
CTB''_j	0	$\frac{3}{4}\csc(\frac{3h}{2})[\cot(\frac{h}{2})+\cot(h)]$	$-\frac{3}{2}\csc(\frac{3h}{2})[\sin(\frac{h}{2})+\cos(h)\csc(\frac{h}{2})]$	$\frac{3}{4}\csc(\frac{3h}{2})[\cot(\frac{h}{2})+\cot(h)]$	0

2.2 Description of the Method:

Consider the self-adjoint second order singularly perturbed boundary value problem of the form

$$lu(x) = -\in u''(x) + a(x)u(x) = f(x) \tag{3}$$

$u(0) = \alpha$

$u(1) = \beta$ (4)

where α and β are constants and $\in$ is a small positive parameter $(0 < \in \leq 1)$, a(x) and f(x) are sufficiently smooth functions. Let a(x)=a= constant and let $u(x) = S(x) = \sum_{j=-1}^{n+1} c_j TB_{j,3}$ is approximate solution of (3). Then let $x_0, x_1, \ldots, x_n$ be n+1 grid points in the interval [0,1]. So we have $x_1=x_0+ih$ where $h=x_{i+1}-x_i=\frac{1}{n}$ at the knot points, and $x_0=0$, $x_n=1$, i=1,2,…,n , we get:

$S(x_i) = \sum_{j=-1}^{n+1} c_j TB_{j,3}(x_i)$

(5)

$S'(x_i) = \sum_{j=-1}^{n+1} c_j TB'_{j,3}(x_i)$

(6)

$S''(x_i) = \sum_{j=-1}^{n+1} c_j TB''_{j,3}(x_i)$

(7)

substituting the value of equations (5) and (7) in equation (3) we get:

$-\in \sum_{j=-1}^{n+1} c_j TB''_{j,3}(x_i) + a(x_i) \sum_{j=-1}^{n+1} c_j TB_{j,3}(x_i) = f(x_i), i = 0,1,2,\ldots,n$ (8)

and the boundary condition becomes,

$\sum_{j=-1}^{n+1} c_j TB_{j,3}(x_0) = \alpha$

(9)

$\sum_{j=-1}^{n+1} c_j TB_{j,3}(x_n) = \beta$

(10)

the values of the spline function at the knot points are determined by using table (1), and substituting in (8)-(10), we get a system of (n+3)× (n+3) equations with (n+3) unknown. Now we write the above system of equations in the following form

$SX_n=I_n$,

where $X_n=(c_{-1}, c_0, \ldots, c_{n+1})^T$ are unknowns

$I_n=(\alpha, \gamma, f(x_0), \ldots, f(x_n), \beta)^T$

Since $TB_{j,3}(x)=\frac{1}{\emptyset}\begin{cases} sin^3\left(\frac{x-x_{i-2}}{2}\right), & [x_{i-2},x_{i-1}] \\ sin^2\left(\frac{x-x_{i-2}}{2}\right)sin\left(\frac{x_i-x}{2}\right)+ & \\ sin\left(\frac{x-x_{i-2}}{2}\right)sin\left(\frac{x_{i+1}-x}{2}\right)sin\left(\frac{x-x_{i-1}}{2}\right) & \\ +sin^2\left(\frac{x-x_{i-1}}{2}\right)sin\left(\frac{x_{i+2}-x}{2}\right), & [x_{i-1},x_i] \\ sin\left(\frac{x-x_{i-2}}{2}\right)sin^2\left(\frac{x_{i+1}-x}{2}\right)+ & \\ sin\left(\frac{x-x_{i-1}}{2}\right)sin\left(\frac{x_{i+1}-x}{2}\right)sin\left(\frac{x_{i+2}-x}{2}\right) & \\ +sin\left(\frac{x-x_i}{2}\right)sin^2\left(\frac{x_{i+2}-x}{2}\right), & [x_i,x_{i+1}] \\ sin^3\left(\frac{x_{i+2}-x}{2}\right) & [x_{i+1},x_{i+2}] \\ 0, & otherwise \end{cases}$

$$TB_{-1,3}(X)=\frac{1}{\emptyset}\begin{cases} sin^3\left(\frac{x_1-x}{2}\right), & [x_0,x_1] \\ 0, & otherwise \end{cases}$$

$$TB_{0,3}(X)=\frac{1}{\emptyset}\begin{cases} sin\left(\frac{x-x_0+2h}{2}\right)sin^2\left(\frac{x_1-x}{2}\right)+ & \\ sin\left(\frac{x-x_0+h}{2}\right)sin\left(\frac{x_1-x}{2}\right)sin\left(\frac{x_2-x}{2}\right) & \\ +sin\left(\frac{x-x_0}{h}\right)sin^2\left(\frac{x_2-x}{2}\right), & [x_0,x_1] \\ sin^3\left(\frac{x_2-x}{2}\right), & [x_1,x_2] \\ 0, & otherwise \end{cases}$$

$$TB_{1,3}(X)=\frac{1}{\emptyset}\begin{cases} sin^3\left(\frac{x-x_0+h}{2}\right), & [x_0-h,x_0] \\ \sin\left(\frac{x-x_0+h}{2}\right)sin^2\left(\frac{x_2-x}{2}\right)+ & \\ sin\left(\frac{x-x_0}{2}\right)sin\left(\frac{x_2-x}{2}\right)sin\left(\frac{x_3-x}{2}\right) & \\ +sin\left(\frac{x-x_1}{2}\right)sin^2\left(\frac{x_3-x}{2}\right), & [x_0,x_1] \\ sin^3\left(\frac{x_2-x}{2}\right), & [x_0,x_1] \\ 0 & otherwise \end{cases}$$

$$T^3B_{n-1}(X)=\frac{1}{\emptyset}\begin{cases} sin^3\frac{(x-x_{n-3})}{2} & [x_{n-3},x_{n-2}] \\ sin\frac{(x-x_{n-3})}{2}sin\frac{(x_{n-1}-x)}{2}+ & \\ sin\frac{(x-x_{n-3})}{2}sin\frac{(x_{n-x})}{2}sin\frac{(x-x_{n-2})}{2} & \\ +sin^2\frac{(x-x_{n-2})}{2}sin\frac{(x_{n+h-x})}{2} & [x_{n-2},x_{n-1}] \\ sin^3\frac{(x_{n+1}-x)}{2} & [x_{n-1},x_n] \\ 0 & otherwise \end{cases}$$

$$T^3B_n(x)=\frac{1}{\emptyset}\begin{cases} sin^3\frac{(x-x_{n-2})}{2} & [x_{n-2},x_{n-1}] \\ sin^2\frac{(x-x_{n-2})}{2}sin\frac{(x_n-x)}{2}+ & \\ sin\frac{(x-x_{n-2})}{2}sin\frac{(x_n+h-x)}{2}sin\frac{(x-x_{n-1})}{2} & \\ +sin^2\frac{(x-x_{n-1})}{2}sin\frac{(x_{n+2h}-x)}{2} & [x_{n-1},x_n] \\ 0 & otherwise \end{cases}$$

$$TB_{n+1,3}(X)=\frac{1}{\emptyset}\begin{cases} sin^3\left(\frac{x-x_{n-1}}{2}\right), & [x_{n-1},x_n] \\ 0, & otherwise \end{cases}$$

Form the boundary conditions equations (9)-(10)) we get :

Since $\sum_{j=-1}^{n+1} C_jT^3B_j(x_0)=\alpha, then$

$$\frac{1}{2}C_{-1}tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+2C_0sin\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+\frac{1}{2}C_1tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)=\alpha \qquad (11)$$

and since $\sum_{j=-1}^{n+1} C_jT^3B_j(x_n)=\beta, then$

$$\frac{1}{2}C_{n-1}tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+2C_nsin\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+\frac{1}{2}C_{n+1}tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)=\beta \quad (12)$$

Also

$$-\in\sum_{j=-1}^{n+1} C_jB_j''(x_i)+a(x_i)\sum_{j=-1}^{n+1} C_jB_j(x_i)=f(x_i)$$

i=0

$$-\in\left(\frac{3}{4}C_{-1}\csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]+\frac{3}{4}C_0\csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]+\frac{3}{4}C_1\csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]\right)+a(x_0)\left(\frac{1}{2}c_{-1}\tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+2c_0\sin\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+\frac{1}{2}c_1\tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)\right)=f(x_0) \qquad (13)$$

i=1

$$-\in\left(\frac{3}{4}C_0\csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]-\frac{3}{2}csc\left(\frac{3h}{2}\right)\left[sin\left(\frac{h}{2}\right)+\cos(h)\,csc\left(\frac{h}{2}\right)\right]\right)+a(x_1)\left(\frac{1}{2}C_0\,tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)\right.$$

$$\left.+2C_1\,sin\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)\right)=f(x_1)$$

(14)

i=2

$$-\in\ \left(\frac{3}{4}C_1\csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]\right)+a(x_2)\left(\frac{1}{2}C_1\tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)\right)=f(x_2) \qquad (15)$$

and i=n-2, then

$$-\in\left(\frac{3}{4}C_{n-1}\csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]\right)+a(x_{n-2})$$
$$\left(\frac{1}{2}C_{n-1}\tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)\right)=f(x_{n-2})$$

(16)

i=n-1

$$-\in\left(-\frac{3}{2}C_{n-1}csc\left(\frac{3h}{2}\right)\left[sin\left(\frac{h}{2}\right)+\cos(h)\,csc\left(\frac{h}{2}\right)\right]+\right.$$
$$\left.\frac{3}{4}C_n\csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]\right)+a(x_{n-1})$$
$$\left(2C_{n-1}sin\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+\frac{1}{2}C_0\tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)\right)=f(x_{n-1}) \qquad (17)$$

i=n,

$$-\in\ \left(\frac{3}{4}C_{n-1}csc\left(\frac{3h}{2}\right)\left[cot\frac{h}{2}+\cot(h)\right]-\frac{3}{2}csc\left(\frac{3h}{2}\right)\left[sin\left(\frac{h}{2}\right)+\cos(h)\,csc\left(\frac{h}{2}\right)\right]+\right.$$
$$\left.\frac{3}{4}C_{n+1}csc\left(\frac{3h}{2}\right)\left[\cot\left(\frac{h}{2}\right)+\cot(h)\right]\right)+a(x_n)\left(\frac{1}{2}C_{n-1}\tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+\right.$$
$$\left.2C_n sin\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)+\frac{1}{2}C_{n+1}tan\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)\right)=f(x_n)$$

(18)

From equation(11)-(18), so the coefficient matrix is given by

$$\left[\begin{array}{ccc}
U & V & U \\
-\frac{3}{4}CSC(\frac{3h}{2}) & 0 & \frac{3}{4}CSC(\frac{3h}{2}) \\
S_0 & T_0 & S_0 \\
0 & S_1 & T_1 \\
 & & S_2 \\
0 & \cdots & \cdots \\
0 & \cdots & \cdots \\
0 & \cdots & \cdots
\end{array}\right.$$

$$\begin{matrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & S_{n-1} & 0 & 0 \\ \vdots & T_{n-1} & S_{n-1} & 0 \\ \vdots & S_n & T_n & S_n \\ 0 & U & V & U \end{matrix}\Bigg] \begin{bmatrix} C_{-1} \\ C_0 \\ \vdots \\ \vdots \\ C_{n-1} \\ C_n \\ C_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_0) \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_n) \\ \beta \end{bmatrix}$$

where

$S_i=-\frac{3}{4}\in\csc(\frac{3h}{2})[\cot(\frac{h}{2})+\cot(h)]+\frac{1}{2}a(x_i)\tan(\frac{h}{2})\csc(\frac{3h}{2})$

$T_i=\frac{3}{2}\in\csc(\frac{3h}{2})\left[\sin\left(\frac{h}{2}\right)+\cos(h)\csc\left(\frac{h}{2}\right)\right]+2a(x_i)\sin\left(\frac{h}{2}\right)\csc(\frac{3h}{2})$

$U=\frac{1}{2}\tan(\frac{h}{2})csc\,\frac{3h}{2})$

$V=2\sin\left(\frac{h}{2}\right)\csc\left(\frac{3h}{2}\right)$ for i=0, 1, 2, n-2, n-1, n

3 Numerical Results

In this section the purpose is the test of the new method for solving ordinary differential equations of boundary value problems through the following example.

Example:

Consider the following second order boundary value problem subject to boundary conditions:

$-\in y''+4y=\frac{(0.3x)^{10}}{625}$,

and boundary conditions y(0)=0 , y(1)=0

∈ \ N	10^{-1}	10^{-2}	10^{-3}
10	$6.648297183\times10^{-10}$	1.994628381×10^{-8}	4.199774960×10^{-9}
20	$4.180808855\times10^{-10}$	3.324979841×10^{-9}	1.953986258×10^{-5}
40	$3.000763563\times10^{-10}$	1.439594520×10^{-9}	2.109720459×10^{-8}

4 Conclusion:

In present chapter we used new method to solve boundary value problem by using of cubic trigonometric B-spline interpolation it seems that the absolute errors are small enough and acceptable; consequently the results were convincing. Moreover, we found that use TBS interpolation, gives more accurate results in comparison with use of B-spline interpolation.

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