http://garmian.edu.krd https://doi.org/10.24271/garmian.125

Localization and Some Properties of Certain Types of Modules

Adil Kadir Jabbar[1] and Rasti Raheem Mohammad Amin[2]
[1]Department of Mathematics, College of Science, University of Sulaimani, Sulaimani, Iraq
[2]Department of Mathematics, College of Education, University of Garmian, Garmian, Iraq
[1]adilkj@gmail.com or adil.jabbar@univsul.edu.iq and [2]rasti.raheem90@gmail.com

Abstract

In this paper Artinian and locally prime modules are studied and some characterizations of locally prime modules are given. Some conditions are given under which locally prime modules are almost prime modules and a nonzero module is a locally prime module. Some properties of Artinian and locally Artinian modules are given. Also, strongly reduced modules, primally reduced modules, radically reduced modules and some other types are studied and investigated and some properties of these types of modules are proved. In addition, some relations that concerning these types of modules are established and some characterizations of them are given.

Keywords: Artinian and locally Artinian modules, locally prime modules, strongly reduced modules, primally reduced modules and radically reduced modules.

1. Introduction

Let M be an $R-$module. A nonempty subset S of R is called a multiplicative system in R, if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [10]. Let M be an $R-$module and N is a submodule of M, the annihilator of N is defined as $Ann(N) = \{r \in M: rN = 0\}$ؤ [11]. As especial case, we have, $Ann(M) = \{r \in R: rM = 0\}$. Let N be a submodule of an $R-$module M, then we define $(N: M)$ as $(N: M) = \{r \in R: rM \subseteq N\}$ [12]. In particular, $(0: M) = \{r \in R: rM = 0\} = Ann(M)$. Let N be a proper submodule of an $R-$module M, then N is called a prime submodule of M, if $r \in R$ and $x \in M$ such that $rx \in N$, then $x \in N$ or $rM \subseteq N$ [4]. Let N be a proper submodule of an $R-$module M, then N is called a semiprime submodule of M, if $r \in R$ and $x \in M$ such that $r^2x \in N$, then $rx \in N$ and M is called a semiprime module if the zero submodule of M is a semiprime submodule [4]. An $R-$module M is called a prime module if the zero submodule of M is a prime submodule of M [2]. An $R-$module M is called an almost prime module if each nonzero proper direct summand of M is a prime submodule of M [4]. An $R-$module M is called a fully prime module if every proper submodule of M is prime and it is called an almost fully prime module, if every non zero proper submodule of M is prime [4]. An $R-$module M is called a fully semiprime module if each proper submodule of M is semiprime and it is called almost fully semiprime if each nonzero proper submodule of M is semiprime [4]. An R −module M is called a locally prime module if M_P is a prime module for each maximal ideal P of R [8]. An R −module M is called an Artinian module if it satisfies the descending chain condition on submodules, equivalently, if there exists a positive integer k such that $N_k = N_{k+n}$, for all $n \in Z^+$ and it is called a locally Artinian module if M_P is an Artinian R_P − module for each maximal ideal P of R [8]. The prime spectrum of an R −module M is denoted by $Spec(M)$ and defined as $Spec(M) = \{N: N$ is a prime submodule of $M\}$ [3]. If N is a submodule of M, then $S_M(N) = \{r \in R: rm \in N$, for some $m \notin N\}$ [1]. A proper submodule N of M is called a primal submodule if $S_M(N)$ forms an ideal of R, this ideal is a proper ideal of R [1]. A proper submodule N of an R −module M is said to be a weakly prime submodule, if whenever $0 \neq rm \in N$, for $r \in R, m \in M$, then $m \in N$ or $rM \subseteq N$ [2]. An R − module M is called a faithful module if $Ann(M) = 0$ ($(0: M) = 0$) [8]. An R −module M is called a cyclic R − module if $M = Rx$, for some $x \in M$ [8]. Let M be an R −module. The primal spectrum of M is denoted by $pSpec(M)$, and is defind as $pSpec(M) = \{N: N$ is a primal submodule of $M\}$ and we say that M is a primally reduced R − module if $\cap pSpec(M) = 0$ [5]. An R − module M is called a reduced module if $\cap Spec(M) = 0$ and it is called locally reduced if M_P is reduced [6]. Let M be an R − module. The Jacobson spectrum of M, denoted by $jpSpec(M) = \{N \in pSpec(M): S_M(N) \subseteq Rad_j(R)\}$, where $Rad_j(R)$ is the Jacobson radical of R and we say that M is radically reduced if $\cap jpSpec(M) = 0$ [5]. Let M be an R − module and P a maximal ideal of R, we define $Spec_p(M) = \{N: N \in Spec(M)$ and $S_M(N) \subseteq P\}$ and we say that M is a strongly reduced R −module if $\cap Spec_p(M) = 0$ **[8]**. A proper submodule N of M is called a maximal submodule if it is not properly contained in any proper submodule of M and the Jacobson radical of R, denoted by $Rad_j(R)$ (or $J(R)$), is defined to be the intersection of all the maximal ideal of R [5]. Let M be an R − module. A submodule A of M is called an essential submodule of M (or M is an essential extension of A), written $A \leq_e M$, if B is any nonzero submodule of M, then $A \cap B \neq \{0\}$, that means every non zero submodule of M must contain at least a non zero element of A. Let M be an R −module and A, B are submodules of M, then the set $\{A, B\}$ is called independent if $A \cap B = \{0\}$. Let M be an R −module and A a submodule of M. A submodule B of M is said to be relative complement for A if $A \cap B = \{0\}$ and B is maximal with respect to the property $A \cap B = \{0\}$(that is, B is not contained properly in any other submodule $\acute{B}$ with the property $\cap \acute{B} = \{0\}$). Let M be an R −module. A(proper) submodule A of M is called a closed submodule of M, written $A \leq_C M$, if A has no proper essential extension in M and if the submodule A is not closed in M, then we write $A \nleq_C M$. A submodule A of an

R −module M is called small (or superfluous), in symbols $A \ll M$, if B is any submodule of M such that $A + B = M$, then $B = M$ (equivalently, if M is the only submodule of M such that $A + M = M$). Let M be an R −module. M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$ [8]. Let M be an R −module. M is called a weak multiplication module if for every prime submodule N of M we have $N = (N: M)M$ [8]. Let R be a commutative ring with identity. Then R it is called a local ring if it has a unique maximal ideal.

2. Artinian and Locally Prime Modules

This section is devoted to study Artinian and locally prime modules. Some characterizations of locally prime modules are given and some conditions are given under which locally prime modules are almost prime modules and also we give a condition which makes a nonzero module as a locally prime module and some properties of Artinian and locally Artinian modules are given.

In the first result we give some characterizations of locally prime modules.

Theorem 2.1. Let M be an R −module. If P is a maximal ideal of R such that $S_M(0) \subseteq P$, then the following conditions are equivalent:

(1) M is a locally prime module.

(2) Each proper direct summand of M is a prime submodule (that is, each nonzero summand becomes a prime module by itself).

(3) All nonzero cyclic submodules of M are isomorphic.

(4) For all $0 \neq m \in M$, we have $Ann(m) = Ann(M)$.

Proof. Since, M is locally prime, so that M_P is a prime module and as $S_M(0) \subseteq P$, the results will follow directly by [9, Theorem 2.10].

The following theorem proves that under certain conditions locally prime modules become almost prime modules.

Theorem 2.2. If M is a locally prime module with $S_M(0) \subseteq P$ and P is a maximal ideal of R, then M is an almost prime R −module.

Proof. Since, M is a locally prime module that means M_P is a prime module, then by [9, Theorem 2.12], we get M is an almost prime R −module.

In the following result we give some conditions under which a nonzero module is locally prime.

Theorem 2.3. If M is a nonzero $R-$ module such that $S_M(0) \subseteq J(R)$ and $S_M(0) \subseteq (0{:}M)$, then M is a locally prime module.

Proof. Let P be any maximal ideal of R, then $S_M(0) \subseteq J(R) \subseteq P$, so by [7, Theorem 2.12], we get $\{0\}$ is a prime submodule of M and by [7, Proposition 2.17], we have $\{0\}_P$ is a proper submodule of M_P. To show $\{0\}_P$ is a prime submodule of M_P. Let $\frac{r}{p}\frac{a}{q} \in \{0\}_P$, where $r \in R, a \in M, p, q \notin P$ and let $\frac{a}{q} \notin \{0\}_P$, then $a \neq 0$, that is $a \notin \{0\}$. Now, $\frac{ra}{pq} \in \{0\}_P$ implies that $tra = 0 \in \{0\}$, for some $t \notin P$. Since, $\{0\}$ is a prime submodule of M, then we get $trM = \{0\}$, then we get $(trM)_P = \{0\}_P$, and as $(trM)_P = \frac{tr}{tp} M_P$, we get $\frac{tr}{tp} M_P = \{0\}_P$, then we get $\frac{r}{p} M_P = \frac{t}{t}\frac{r}{p} M_P = \frac{tr}{tp} M_P = \{0\}_P$. Hence, $\{0\}_P$ is a prime submodule of M_P, so that M_P is a prime module that means M is a locally prime module.

The following result shows that under certain conditions the localization of a prime submodule is prime.

Theorem 2.4. Let M be a locally Artinian $R-$ module. If P is a maximal ideal of R and N is a primal submodule of M with $S_M(N) \subseteq P$ and $(N:M)$ is a maximal ideal of R, then N_P is a prime submodule of M.

Proof. As N is primal and $(N:M)$ is a maximal ideal of R, by [8, Proposition 2.24], we get N is a prime submodule of M and as $S_M(N) \subseteq P$, so by [8, Proposition 2.21], we get N_P is a prime submodule of M.

Theorem 2.5. Let M be an $R-$ module, N a proper submodule of M with $S_M(N) \subseteq P$ and P a maximal ideal of R. If N_P is a prime submodule of M_P, then $(N:M)$ is a maximal ideal of R.

Proof. Since, N_P is a prime submodule of M_P and $S_M(N) \subseteq P$, so by [8, Proposition 2.21], we get N is a prime submodule of M and then by [8, Proposition 2.23], we get $(N:M)$ is a maximal ideal of R.

Theorem 2.6. Let M be an Artinian R −module and P a maximal ideal of R. If N is a proper submodule of M such that $S_M(N) \subseteq P$, then N_P is a prime submodule of M_P if and only if $(N:M)$ is a maximal ideal of R.

Proof. ($\Rightarrow$) Let N_P be a prime submodule of M_P. Since we have, $S_M(N) \subseteq P$, so by [8, Proposition 2.21], we get N is a prime submodule and then by [3, Corollary 2.4], we get $(N:M)$ is a maximal ideal of R.

($\Leftarrow$) Let $(N:M)$ be a maximal ideal of R, then by [3, Corollary 2.4], we get N is a prime submodule and as $S_M(N) \subseteq P$, by [8, Proposition 2.21], we get N_P is a prime submodule of M_P.

In the following result we give some conditions which make the localization of a locally Artinian module as a prime module.

Theorem 2.7. Let M be a locally Artinian R −module and P a maximal ideal of R such that $S_M(0) \subseteq J(R)$. If $Ann(M)$ is a primal ideal of R, then M_P is a prime R_P −module if and only if $R/(Ann(M))$ is a field.

Proof. ($\Rightarrow$) Let M_P be a prime module. Now, we have $S_M(0) \subseteq J(R) \subseteq P$, thus by [8, Proposition 2.14], we get M is a prime module and by [8, Proposition 2.17], we have $R/(Ann(M))$ is a field.

($\Leftarrow$) As $S_M(0) \subseteq J(R)$, we have $Ann(M)$ is a primal ideal of R and $R/(Ann(M))$ is a field and by [8, Proposition 2.17], we get M is a prime module, then by [9, Theorem 2.11], we get M_P is a prime R_P −module.

The next result proves that under certain condition the localization of a locally prime module is a prime module.

Theorem 2.8. Let M be an $R -$ module. If M is locally prime and $S_M(0) \subseteq J(R)$, then M_P is a prime module.

Proof. By [8, Corollary 2.15], we get M is a prime module and by [9, Theorem 2.11], we have M_P is prime.

In the following result we give some conditions under which we can characterize those faithful locally Artinian modules the localization of which are prime.

Theorem 2.9. Let M be a faithful localy Artinian $R -$ module and P be a maximal ideal of R. If R is a primal ring and $S_M(0) \subseteq J(R)$, then M_P is a prime module if and only if R is a field.

Proof ($\Rightarrow$) Let M_P be a prime and P be a maximal ideal of R, then $S_M(0) \subseteq J(R) \subseteq P$, thus by [8, Proposition 2.14], we get M is prime and by [8, Corollary 2.19], we have R is a field.

($\Leftarrow$) By [8, Corollary 2.19], we get M is prime and by [9, Theorem 2.11], we have M_P is a prime.

Theorem 2.10. Let M be an $R -$ module and N be a proper submodule of M. If P is a maximal ideal of R such that $S_M(N) \subseteq P$ and the DCC is satisfied on cyclic submodules of M_P, then N_P is a prime submodule of M_P if and only if N_P is a weakly prime submodule of M_P.

Proof. ($\Rightarrow$) Let N_P be a prime submodule of M_P, then by [8, Proposition 2.21], we get N is a prime submodule of M, so by [8, Corollary 2.22], we get N is a weakly prime submodule of M and by [8, Proposition 2.21], we have N_P is a weakly prime submodule of M_P.

($\Leftarrow$) Let N_P be a weakly prime submodule of M_P, then by [8, Proposition 2.21], we get N is a weakly prime submodule of M, so by [8, Corollary 2.22], we get N is a prime submodule of M and by [8, Proposition 2.21], we have N_P is a prime submodule of M_P.

In the following two results we give some further conditions under which the localization of (faithful) Artinian modules are prime.

Theorem 2.11. Let M ba an Artinian $R -$ module and P ba a maximal ideal of R such that $S_M(0) \subseteq P$, then M_P is a prime module if and only if $R/(Ann(M))$ is a field.

Proof ($\Rightarrow$) Let M_P be a prime module and $S_M(0) \subseteq P$, then by [8, Proposition 2.14], we get M is a prime module and by [3, Proposition 2.1], we have $R/(Ann(M))$ is a field.

($\Leftarrow$) By [3, Proposition 2.1], we have M is a prime module and by [9, Theorem 2.11], we get M_P is a prime module.

Theorem 2.12. Let M be a faithful Artinian $R-$ module and P ba a maximal ideal of R such that $S_M(0) \subseteq P$, then M_P is a prime module if and only if R is a field.

Proof. ($\Rightarrow$) Let M_P be a prime module. As, $S_M(0) \subseteq P$, by [8, Proposition 2.14], we get M is a prime module and by [3, Corollary 2.2], we have R is a field.

($\Leftarrow$) Let R be a field. By [3, Corollary 2.2], we have M is a prime module and by [9, Theorem 2.11], we get M_P is a prime module.

3. Strongly Reduced, Primally Reduced and Radically Reduced Modules

In this section, further types of modules are studied and investigated such as, strongly reduced modules, primally reduced modules, radically reduced modules and some other types. Some properties of these types of modules are proved and some relations between them are determined and also some characterizations of them are given.

In the first result we prove that under certain condition, if the localization of a module is reduced, then the module itself is also reduced.

Theorem 3.1. Let M be an $R-$module and P be a maximal ideal of R such that $S_M(0) \subseteq P$. If M_P is a reduced R_P-module, then M is a reduced $R-$module.

Proof. Let $x \in \cap Spec(M)$. Let $\bar{N} \in Spec(M_P)$, so that $\bar{N}$ is prime submodule of M_P. Then by [8, Lemma 2.27], we have $\bar{N} = N_P$ for the prime submodule $N = \{x \in M: \frac{x}{1} \in \bar{N}\}$ of M with $S_M(N) \subseteq P$, that means N_P is a prime submodule of M_P, then by [8, Lemma 2.27], we get N is a prime submodule of M, that means $N \in Spec(M)$, so we get $x \in N$ and then $\frac{x}{1} \in N_P = \bar{N}$, thus we get $\frac{x}{1} \in \cap Spec(M_P)$, but $\cap Spec(M_P) = 0$ we get $\frac{x}{1} = 0$ and as $S_M(0) \subseteq P$, by [8, Lemma 2.1], we get $x = 0$, so we have $\cap Spec(M) = 0$. That means M is a reduced module.

The next result shows that the localization of strongly reduced modules are also strongly reduced.

Theorem 3.2. Let M be an $R-$module and P be a maximal ideal of R. If M is strongly reduced, then M_P is strongly reduced.

Proof. Let $\frac{x}{p} \in \cap Spec_P(M_P)$, where $x \in M$ and $p \notin P$. Let $N \in \cap Spec_P(M)$, then $N \in Spec(M)$ and $S_M(N) \subseteq P$, so by [8, Proposition 2.20], we get $S_{M_P}(N_P) = (S_M(N))_P \subseteq P_P$, that means $S_{M_P}(N_P) \subseteq P_P$ and then by [8, Proposition 2.21], we get N_P is a prime submodule of M_P, that means $N_P \in Spec(M_P)$ and $S_{M_P}(N_P) \subseteq P_P$, we get $N_P \in Spec_P(M_P)$, so that $\frac{x}{p} \in N_P$ and by [8, Lemma 2.1], we have $x \in N$, thus we get $x \in \cap Spec_P(M)$, but $\cap Spec_P(M) = 0$ we get $x = 0$ and

$p \notin P$ then $\frac{x}{p} = \frac{0}{p} = 0$, so we have $\cap Spec_P(M_P) = 0$, that means M_P is strongly reduced.

In the following result we give a condition under which the converse of the last theorem is true.

Theorem 3.3. Let M be an $R-$module and P be a prime ideal of R such that $S_M(0) \subseteq P$. If M_P is a strongly reduced R_P $-$module, then M is a strongly reduced module.

Proof. Let $x \in \cap Spec_P(M)$ and let $\bar{N} \in Spec_P(M_P)$, that is $\bar{N}$ is prime submodule of M_P and $S_{M_P}(\bar{N}) \subseteq P_P$. Then by [8, Lemma 2.27], we have $\bar{N} = N_P$ for the prime submodule $N = \{x \in M: \frac{x}{1} \in \bar{N}\}$ of M with $S_M(N) \subseteq P$, that means N_P is a prime submodule of M_P and $S_{M_P}(N_P) \subseteq P_P$, that means $N \in Spec_P(M)$, so we get $x \in N$ and $p \notin P$, then we have $\frac{x}{p} \in N_P = \bar{N}$, thus we get $\frac{x}{p} \in \cap Spec_P(\bar{N})$ and as $\cap Spec_P(\bar{N}) = 0$, we get $\frac{x}{p} = 0$, then by [8, Lemma 2.1], we get $x = 0$, thus we have $\cap Spec_P(M) = 0$. That means, M is strongly reduced.

Next we prove that, under a certain condition those modules localization of which are strongly reduced are reduced.

Corollary 3.4. If M_P is strongly reduced and P be a maximal ideal of R such that $S_M(0) \subseteq P$, then M is reduced.

Proof. Since, M_P is a strongly reduced R_P $-$module, so by Theorem 3.3, we get M is a strongly reduced $R-$module, that gives $\cap Spec_P(M) = 0$. Then by [6, Theorem 2.4], we get $\cap Spec(M) = 0$.

Theorem 3.5. Let M be an R $-$module and P a maximal ideal of R, then we have $(\cap Spec_P(M))_P = \cap Spec_P(M_P)$.

Proof. Let $\frac{x}{p} \in (\cap Spec_P(M))_P$, for $x \in M$ and $p \notin P$. Then $qx \in \cap Spec_P(M)$, for some $q \notin P$. Let $\bar{N} \in Spec_P(M_P)$, that is $\bar{N}$ is prime submodule of M_P. Then by [8, Lemma 2.27], we have $\bar{N} = N_P$ for the prime submodule $N = \{x \in M: \frac{x}{1} \in \bar{N}\}$ of M with $S_M(N) \subseteq P$, so that $N \in Spec_P(M)$ and thus $qx \in N$, from which we get $\frac{x}{p} = \frac{q}{q}\frac{x}{p} = \frac{qx}{qp} \in N_P = \bar{N}$ and so $\frac{x}{p} \in \cap Spec_P(M_P)$ and thus we have $(\cap Spec_P(M))_P \subseteq \cap Spec_P(M_P)$. Now, let $\frac{x}{p} \in \cap Spec_P(M_P)$, where $x \in M$ and $p \notin P$. Let $N \in Spec_P(M)$, so that $N \in Spec(M)$ and $S_M(N) \subseteq P$, then by [8, Proposition 2.20], we get $S_{M_P}(N_P) = (S_M(N))_P \subseteq P_P$, that means $S_{M_P}(N_P) \subseteq P_P$ and by [8, Proposition 2.21], we get N_P is a prime submodule of M_P, that is $N_P \in Spec_P(M_P)$, so that $\frac{x}{p} \in N_P$ and by [8, Lemma 2.1], we have $x \in N$, thus we get $x \in \cap Spec_P(M)$, this gives $\frac{x}{p} \in (\cap Spec_P(M))_P$, so that $\cap Spec_P(M_P) \subseteq (\cap Spec_P(M))_P$. Hence, we get $(\cap Spec_P(M))_P = \cap Spec_P(M_P)$.

The following corollary proves that, if the localization of a module is strongly reduced, then this localization is reduced.

Corollary 3.6. Let M be an $R-$module and P a maximal ideal of R. If M_P is a strongly reduced R_P-module, then M_P is a reduced R_P-module.

Proof. Since, M_P is strongly reduced so that $\bigcap Spec_P(M_P) = 0$. Then by Theorem 3.5, we get $(\bigcap Spec_P(M))_P = 0$ and by [6, Theorem 2.4], we get $\bigcap Spec(M_P) = 0$, that means, M_P is reduced.

Next, we prove that the localization of strongly reduced modules are reduced.

Corollary 3.7. Let M be an $R-$module and P a maximal ideal of R. If M is a strongly reduced $R-$module, then M_P is a reduced R_P-module.

Proof. Since, M is strongly reduced, so by Theorem 3.2, we get M_P is a strongly reduced R_P- module that is $\bigcap Spec_P(M_P) = 0$. By Theorem 3.5, we have $(\bigcap Spec_P(M))_P = \bigcap Spec_P(M_P) = 0$ and then by [6, Theorem 2.4], we get $\bigcap Spec(M_P) = 0$.

In the following theorem, we prove that the localization of radically reduced modules are radically reduced.

Theorem 3.8. Let M be an $R-$module and P be a maximal ideal of R. If M is radically reduced, then M_P is radically reduced.

Proof. Let $\frac{x}{p} \in \bigcap jpSpec(M_P)$, where $x \in M$ and $p \notin P$. Let $N \in \bigcap jpSpec(M)$, then $N \in pSpec(M)$ and $S_M(N) \subseteq Rad_j(R) \subseteq P$. By [5, Proposition 2.5], N_P is a primal submodule of M_P, that is $N_P \in pSpec(M_P)$, so that $S_{M_P}(N_P)$ is a proper ideal of R_P and since, R_P is a local ring with the unique maximal ideal P_P, so that $S_{M_P}(N_P) \subseteq P_P = Rad_j(R_P)$, so that $\frac{x}{p} \in N_P$ and then by [8, Lemma 2.1], we have $x \in N$, thus we get $x \in \bigcap jpSpec(M)$, but $\bigcap jpSpec(M) = 0$ we get $x = 0$ and $p \notin P$ then $\frac{x}{p} = \frac{0}{p} = 0$, we have $\bigcap jpSpec(M_P) = 0$ that means M_P is radically reduced.

Now, for the local rings we give a condition which makes the converse of the theorem is also true.

Theorem 3.9. Let R be a local ring with P as its unique maximal ideal and M be an $R-$module such that $S_M(0) \subseteq P$. If M_P is radically reduced, then M is radically reduced.

Proof. Let $x \in \bigcap jpSpec(M)$ and $\bar{N} \in jpSpec(M_P)$, that is $\bar{N}$ is primal submodule of M_P. Then by [5, Proposition 2.6], we have $\bar{N} = N_P$ for the primal submodule $N = \{x \in M: \frac{x}{1} \in \bar{N}\}$ of M with $S_M(N) \subseteq P$, that means N_P is a primal submodule of M_P and $S_{M_P}(N_P) \subseteq Rad_j(R_P) = P_P$ and by [5, Proposition 2.6], we get N is a primal submodule of M, so that $N \in pSpec(M)$ and $S_M(N)$ is a (proper) ideal of R and thus we get $S_M(N) \subseteq P(= Rad_j(R))$, which means that $N \in jpSpec(M)$, then

$x \in N$ and $p \notin P$, we have $\frac{x}{p} \in N_P$, thus we get $\frac{x}{p} \in \bigcap jpSpec(N_P)$, but $\bigcap jpSpec(N_P) = 0$ we get $\frac{x}{p} = 0$, so by [8, Lemma 2.1], we get $x = 0$, then we have $\bigcap jpSpec(M) = 0$. That means M is a radically reduced R −module.

Next, we determine a relation between the primal spectrum and the Jacobson radical of a module.

Theorem 3.10. Let M be an R − module and let P be a maximal ideal of R, then $\bigcap pSpec(M) \subseteq \bigcap jpSpec(M)$.

Proof. As, $jpSpec(M) \subseteq pSpec(M)$, we get $\bigcap pSpec(M) \subseteq \bigcap jpSpec(M)$.

The following theorem shows that the localization of the Jacobson radical of a module and the Jacobson radical of the localization are the same.

Theorem 3.11. Let R be a local ring with P as its unique maximal ideal and M be an R −module, then we have $(\bigcap jpSpec(M))_P = \bigcap jpSpec(M_P)$.

Proof. Let $\frac{x}{p} \in (\bigcap jpSpec(M))_P$, where $x \in M$ and $p \notin P$. Then, there exists $q \notin P$ such that $qx \in \bigcap jpSpec(M)$. Now, let $\bar{N} \in jpSpec(M_P)$, that is $\bar{N}$ is primal submodule of M_P. Then by [5, Proposition 2.6], we have $\bar{N} = N_P$ for the primal submodule $N = \{x \in M: \frac{x}{1} \in \bar{N}\}$ of M with $S_M(N) \subseteq P$, that is $N_P \in pSpec(M_P)$ and $S_{M_P}(N_P) \subseteq Rad_j(R_P) = P_P$ and by [5, Proposition 2.6], we get N is a primal submodule of M, so that $N \in pSpec(M)$ and $S_M(N)$ is a (proper) ideal of R and thus we get $S_M(N) \subseteq P(= Rad_j(R))$, which means that $N \in jpSpec(M)$ and thus $qx \in N$, from which we get $\frac{x}{p} = \frac{q}{q}\frac{x}{p} = \frac{qx}{qp} \in N_P = \bar{N}$, so that $\frac{x}{p} \in \bigcap jpSpec(M_P)$ and thus we have $(\bigcap jpSpec(M))_P \subseteq \bigcap jpSpec(M_P)$. Now, let $\frac{x}{p} \in \bigcap jpSpec(M_P)$, where $x \in M$ and $p \notin P$. Let $N \in jpSpec(M)$, then $N \in pSpec(M)$ and $S_M(N) \subseteq Rad_j(R) \subseteq P$. By [5, Proposition 2.5], N_P is a primal submodule of M_P, that is $N_P \in pSpec(M_P)$, so that $S_{M_P}(N_P)$ is a proper ideal of R_P and since, R_P is a local ring with the unique maximal ideal P_P, so that $S_{M_P}(N_P) \subseteq P_P = Rad_j(R_P)$, we get $N_P \in jpSpec(M_P)$, so that $\frac{x}{p} \in N_P$ and then by [8, Lemma 2.1], we have $x \in N$, thus we get $x \in \bigcap jpSpec(M)$, this gives $\frac{x}{p} \in (\bigcap jpSpec(M))_P$, so that $\bigcap jpSpec(M_P) \subseteq (\bigcap jpSpec(M))_P$. Hence, we get $(\bigcap jpSpec(M))_P = \bigcap jpSpec(M_P)$.

The following corollary shows that, radically reducedness property implies primally reducedness for the localized module.

Corollary 3.12. Let R be a local ring with P as its unique maximal ideal and M be an R −module. If M_P is radically reduced, then M_P is primally reduced.

Proof. Since, M_P is a radically reduced, so that $\bigcap jpSpec(M_P) = 0$. Then, by Theorem 3.11, we get $(\bigcap jpSpec(M))_P = 0$ and by [5, Theorem 2.11], we get

$\cap pSpec(M_P) \subseteq \left(\cap jpSpec(M)\right)_P = 0$ that means M_P is a primally reduced R_P −module.

In the following corollary, we give a condition which makes those modules the localization of which are radically reduced are primally reduced.

Corollary 3.13. Let R be a local ring with P as its unique maximal ideal and M be an R −module such that $S_M(0) \subseteq P$. If M_P is radically reduced, then M is primally reduced.

Proof. By Theorem 3.9, we have M is a radically reduced and by [5, Corollary 2.12], we get M is a primally reduced R −module.

Corollary 3.14. Let M be a multiplication and a locally reduced $R-$ module and N be a submodule of M. If M_P is a primally reduced $R-$ module, then $N \cap Ann(N)M = 0$.

Proof. By [5, Corollary 2.9], we have M is primally reduced and by [5, Proposition 2.17], we get $N \cap Ann(N)M = 0$.

In the following theorem, we give a condition which makes reduced modules, radically reduced modules and primally reduced modules equivalent.

Theorem 3.15. Let R be a local ring with P as its uniqe maximal ideal and M be an R −module such that $S_M(0) \subseteq P$. The following statements are equivalent:

(1) M_P is radically reduced.

(2) M is primally reduced.

(3) M is reduced.

Proof. ($1 \Rightarrow 2$) By Corollary 3.13, we have M is a primally reduced R −module.

($2 \Rightarrow 3$) By [5, Theorem 2.16], we get M is a reduced R −module.

($3 \Rightarrow 1$) By [5, Theorem 2.16], we get M is radically reduced and by Theorem 3.8, we get M_P is radically reduced.

By assuming some conditions in the following theorem, we give a necessary and sufficient condition for a submodule to have a weakly prime localization.

Theorem 3.16. Let M be an R −module and N a proper submodule of M with $S_M(0) \subseteq (N:M)$. If P is a maximal ideal of R such that $S_M(N) \subseteq P$, then N_P is weakly prime if and only if $S_M(N) \subseteq (N:M)$.

Proof. ($\Rightarrow$) Let N_P be weakly prime, then by [8, Proposition 2.21], we get N is weakly prime and then by [5, Proposition 2.15], we get $S_M(N) \subseteq (N:M)$.

($\Leftarrow$) suppose that $S_M(N) \subseteq (N:M)$. By [5, Proposition 2.15], we get N is weakly prime and by [8, Proposition 2.21], we get N_P is weakly prime.

Next, we give some conditions which make the modules that have weak multiplication localization as weak multiplication modules.

Theorem 3.17. Let M be an R −module and N a proper submodule of M with $S_M(N) \subseteq J(R)$. If P is a maximal ideal of R such that M_P is a weak multiplication R_P −module, then M is a weak multiplication R −module.

Proof. Let P be any maximal ideal of R, then $S_M(N) \subseteq J(R) \subseteq P$. If N ia any prime submodule of M, then by [8, Proposition 2.21], we get N_P is a prime submodule of M_P and as M_P is a weak multiplication module, we have $N_P = (N_P: M_P)M_P$, then by [7, Theorem 2.21], we get $N_P = ((N:M)M)_P$, so by [7, Corollary 2.2], we get $N = (N:M)M$, that means M is a weak multiplication R −module.

In the following theorem, we prove that the localization of prime and regular modules are fully prime.

Theorem 3.18. Let M be an R −module and P be a prime ideal of R. If M is a prime and regular module, then M_P is a fully prime R_P −module.

Proof. By [4, Corollary 1.9], we get M is a fully prime module and by [9, Theorem 2.1], we get M_P is a fully prime R_P −module.

Theorem 3.19. If each cyclic submodule of an R −module M is a prime submodule and P is a prime ideal of R, then M_P is prime and each cyclic submodule of M is semiprime.

Proof. By [4, Corollary 1.9], we get M is prime and each cyclic submodule of M is semiprime and by [9, Theorem 2.11], we get M_P is prime.

Theorem 3.20. Let M be an R −module and N be a proper submodule of M. If P is a maximal ideal of R with $S_M(N) \subseteq P$ and $(N:M)$ is a maximal ideal of R, then $(N:M)_P$ is a maximal ideal of R_P.

Proof. Since $1 \notin P$, implies that $\frac{1}{1}$ is the identity of R_P and $(N:M)_P = R_P$, let $\frac{1}{1} \in (N:M)_P = (N_P: M_P)$, implies that $\frac{1}{1} M_P \subseteq N_P$, we get $M_P \subseteq N_P$, that means $M_P = N_P$, which is a contradiction. To show that $(N:M)_P$ is a maximal ideal of R_P, so let $(N:M)_P \subseteq \bar{J} \subseteq R_P$, for the ideal $\bar{J}$ of R_P. By [7, Proposition 2.16], we have $\bar{J} = J_P$, for the ideal $J = \{a \in R: \frac{a}{1} \in \bar{J}\}$ of R, so that $(N:M)_P \subseteq J_P \subseteq R_P$. Suppose that $J_P \neq R_P$, so that $J \neq R$. If $a \in (N:M)$, then $\frac{a}{1} \in J_P$, so $pa \in J$, for some $p \notin P$, then $\frac{pa}{1} \in \bar{J}$. Now, $\frac{a}{1} = \frac{1}{p}\frac{pa}{1} \in \bar{J}$, so that $a \in J$ and thus $(N:M) \subseteq J \subseteq R$. As $(N:M)$ is a maximal, we get $(N:M) = J$, so that $(N:M)_P = J_P$, that means $(N:M)_P$ is a maximal ideal of R_P.

Theorem 3.21. Let M be a locally Artinian R −module and N a proper submodule of M. If P is a maximal ideal of R with $S_M(N) \subseteq P$ and $(N:M)_P$ is a maximal ideal of R_P, then $(N:M)$ is a maximal ideal of R.

Proof. As N is proper, by [7, Proposition 2.17],we get N_P is a proper submodule of M_P and then by [7, Theorem 2.21], we have $(N:M)_P = (N_P: M_P)$ and Since, $(N:M)_P$ is a maximal ideal of R_P and M_P is an Artinian module, by [3, Corollary 2.4], we have N_P is a prime submodule of M_P and by [8, Proposition 2.21], we get N is a prime submodule of M and by [8, Proposition 2.23], we get $(N:M)$ is a maximal ideal of R.

References:

[1] Atani, S. E. and Darani, A. Y. : Notes on the Primal Submodules, Chiang Mai J. Sci.
35(3), 2008, pp 399-410.

[2] Atani, S. E. and Farzalipour, F. : On Weakly Prime Submodules, Tamkang Journal of Mathematics, Vol. 38, No. 3, 2007, 247-252.

[3] Azizi, A.: Prime Submodules of Artinian Modules, Taiwaness Journal of Mathematics, Vol. 13, No. 6B, pp. 2011-2020, 2009.

[4] Behboodi, M. , Karamzadeh, O.A.S. and Koohy, H. : Modules Whose Certain Submodules Are Prime, Vietnam Journal of Mathematics 32:3 (2004) pp 303-307.

[5] Jabbar, A. K.: A Generalization of Reduced Modules, International Journal of Algebra , Vol. 8, 2014, no. 1, 39-45.

[6] Jabbar, A. K.: On Locally Reduced and Locally Multiplication Modules, International Mathematical Forum, Vol. 8, 2013, no. 18, 851-858,

[7] Jabbar, A. K.: A Generalization of prime and weakly prime submodules, Pure Mathematical Sciences, Vol. 2, 2013, No. 1, 1-11 ,

[8] Jabbar, A. K.: On Locally Artinian Modules, International Journal of Algebra, Vol. 6, 2012, No. 27, 1325-1334.

[9] Jabbar, A. K. and Mohammad R. R.: The Effect of Localization on Properties of Certain Types of Modules, The 2nd International conference of the college of Education – University of Garmian held on 21-22/8/2016.

[10] Larsen, M. D. and McCarthy, P. J.: Multiplicative Theory of Ideals, Academic Press, New York and London, 1971.

[11] Lomp, C. and Pena, P. A. J. : A Note on Prime Modules, Divulgaciones Mathematicas, Vol. 8, No. 1, 2000, pp 31-42.

[12] Rajaee, S. : Comaximal Submodules of Multiplication Modules, International Mathematical Forum, Vol.5, 2010, no. 24, 1179-1183.