On Minimal $\lambda_{ac}$-Open Sets

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Abstract

In this paper, we introduce and discuss minimal $\lambda_{ac}$-open sets in topological spaces. We establish some basic properties of minimal $\lambda_{ac}$-open. We obtain an application of a theory of minimal $\lambda_{ac}$-open sets and we defined a $\lambda_{ac}$-locally finite space.

1. Introduction

The study of semi open sets in topological spaces was initiated by Levine[1]. The complement of $A$ is denoted by $X \setminus A$. In the space $(X, \tau)$, a subset $A$ is said to be $\alpha$-open [2] if $A \subseteq Int(Cl(Int(A)))$. The family of all $b$-open sets of $(X, \tau)$ is denoted by $BO(X)$. The complement of $\alpha$-open is called $\alpha$-closed. The concept of operation $\gamma$ was initiated by Kasahara [3]. He also introduced $\gamma$-closed graph of a function. Using this operation, Ogata[4] introduced the concept of $\gamma$-open sets and investigated the related topological properties of the associated topology $\tau_{\gamma}$ and $\tau$. He further investigated general operator approaches of closed graph of mappings. Further Ahmad and Hussain[5] continued studying the properties of $\gamma$-open($\gamma$-closed) sets. In 2009, Hussain and Ahmad [6], introduced the concept of minimal $\gamma$-open sets. In 2011[7] (resp., in 2013[8]) Khalaf and Namiq defined an operation $\lambda$ called s-operation. They defined $\lambda^*$-open sets [9] which is equivalent to $\lambda$-open set[7] and $\lambda_s$-open set[8] by using s-operation. They work in operation in topology in [10-22]. They defined $\lambda_{\beta c}$-open set by using s-operation and $\beta$-closed set and also investigated several properties of $\lambda_{\beta c}$-derived, $\lambda_{\beta c}$-interior and $\lambda_{\beta c}$-closure points in topological spaces.

In this paper, we introduce and discuss minimal $\lambda_{ac}$-open sets in topological spaces. We establish some basic properties of minimal $\lambda_{ac}$-open sets and provide an example to illustrate that minimal $\lambda_{ac}$-open sets are independent of minimal open sets.

First, we recall some definitions and results used in this paper.

2. Preliminaries

Throughout, $X$ denotes a topological space. Let $A$ be a subset of $X$, then the closure and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset $A$ of a topological space $(X, \tau)$ is said to be semi open [1] if $A \subseteq Cl(Int(A))$. The complement
of a semi open set is said to be semi closed [1]. The family of all semi open (resp. semi closed) sets in a topological space \((X,\tau)\) is denoted by \(SO(X,\tau)\) or \(SO(X)\) (resp. \(SC(X,\tau)\) or \(SC(X)\)). We consider \(\lambda\) as a function defined on \(SO(X)\) into \(P(X)\) and \(\lambda: SO(X) \rightarrow P(X)\) is called an \(s\)-operation if \(V \subseteq \lambda(V)\) for each non-empty semi open set \(V\). It is assumed that \(\lambda(\phi) = \phi\) and \(\lambda(X) = X\) for any \(s\)-operation \(\lambda\). Let \(X\) be a topological space and \(\lambda: SO(X) \rightarrow P(X)\) be an \(s\)-operation, then a subset \(A\) of \(X\) is called a \(\lambda^*\)-open set [9]which is equivalent to \(\lambda\) –open set[7] and \(\lambda_s\)-open set [8] if for each \(x \in A\) there exists a semi open set \(U\) such that \(x \in U\) and \(\lambda(U) \subseteq A\).

The complement of a \(\lambda^*\)-open set is said to be \(\lambda^*\)-closed. The family of all \(\lambda^*\)-open (resp., \(\lambda^*\)-closed ) subsets of a topological space \((X,\tau)\) is denoted by \(SO_{\lambda}(X,\tau)\) or \(SO_{\lambda}(X)\) (resp., \(SC_{\lambda}(X,\tau)\) or \(SC_{\lambda}(X)\)).

**Definition 2.1.** A \(\lambda^*\)-open[9]( \(\lambda\) -open[7], \(\lambda_s\)-open[8] ) subset \(A\) of a topological space \(X\) is called \(\lambda_{\beta c}\)-open [23] if for each \(x \in A\) there exists a \(\beta\)-closed set \(F\) such that \(x \in F \subseteq A\). The complement of a \(\lambda_{\beta c}\)-open set is called \(\lambda_{\beta c}\)-closed[23]. The family of all \(\lambda_{\beta c}\)-open (resp., \(\lambda_{\beta c}\)-closed) subsets of a topological space \((X,\tau)\) is denoted by \(SO_{\lambda_{\beta c}}(X,\tau)\) or \(SO_{\lambda_{\beta c}}(X)\)( resp. \(SC_{\lambda_{\beta c}}(X,\tau)\) or \(SC_{\lambda_{\beta c}}(X)\) ) [23].

We get the following results in [23]

**Proposition 2.2.** For a topological space \(X\), \(SO_{\lambda_{\beta c}}(X) \subseteq SO_{\lambda}(X) \subseteq SO(X)\).

The following example shows that the converse of the above proposition may not be true in general.

**Example 2.3.** Let \(X = \{a,b,c\}\), and \(\tau = \{\phi,\{a\},X\}\). We define an \(s\)-operation \(\lambda: SO(X) \rightarrow P(X)\) as \(\lambda(A) = A\) if \(b \in A\) and \(\lambda(A) = X\) otherwise. Here, we have \(\{a,c\}\) is semi open but it is not \(\lambda^*\)-open. And also \(\{a,b\}\) is \(\lambda^*\)-open set but it is not \(\lambda_{ac}\)-open.

**Definition 2.4.** An \(s\)-operation \(\lambda\) on \(X\) is said to be \(s\)-regular which is equivalent to \(\lambda\) -regular [8]if for every semi open sets \(U\) and \(V\) of \(x \in X\), there exists a semi open set \(W\) containing \(x\) such that \(\lambda(W) \subseteq \lambda(U) \cap \lambda(V)\).

**Definition 2.5.** Let \(A\) be a subset of \(X\). Then:

1. The \(\lambda_{\beta c}\)-closure of \(A\) \((\lambda_{\beta c} Cl(A))\) is the intersection of all \(\lambda_{\beta c}\)-closed sets containing \(A\).
2. The \(\lambda_{\beta c}\)-interior of \(A\) \((\lambda_{\beta c} Int(A))\) is the union of all \(\lambda_{\beta c}\)-open sets of \(X\) contained in \(A\).

**Proposition 2.6.** For each point \(x \in X\), \(x \in \lambda_{\beta c} Cl(A)\) if and only if \(V \cap A \neq \phi\) for every \(V \in SO_{\lambda_{\beta c}}(X)\) such that \(x \in V\).
Proposition 2.7. Let \( \{A_\alpha\}_{\alpha \in I} \) be any collection of \( \lambda_{\beta c} \)-open sets in a topological space \((X, \tau)\), then \( \bigcup_{\alpha \in I} A_\alpha \) is a \( \lambda_{\beta c} \)-open set.

Proposition 2.8. Let \( \lambda \) bean \( s \)-regular \( s \)-operation. If \( A \) and \( B \) are \( \lambda_{\beta c} \)-open sets in \( X \), then \( A \cap B \) is also a \( \lambda_{\beta c} \)-open set.

The proof of the following two propositions are in [24].

Proposition 2.9. Let \( \{A_\alpha\}_{\alpha \in I} \) be any collection of \( \lambda^* \)-open sets in a topological space \((X, \tau)\), then \( \bigcup_{\alpha \in I} A_\alpha \) is a \( \lambda^* \)-open set.

Proposition 2.10. Let \( \lambda \) be semi-regular operation. If \( A \) and \( B \) are \( \lambda^* \)-open sets in \( X \), then \( A \cap B \) is also a \( \lambda^* \)-open set.

Definition 2.11. A \( \lambda^* \)-open[9] (\( \lambda \)-open[7], \( \lambda_s \)-open[8]) subset \( A \) of a topological space \( X \) is called \( \lambda_{ac} \)-open if for each \( x \in A \) there exists a \( b \)-closed set \( F \) such that \( x \in F \subseteq A \).

The complement of a \( \lambda_{ac} \)-open set is called \( \lambda_{ac} \)-closed. The family of all \( \lambda_{ac} \)-open (resp., \( \lambda_{ac} \)-closed) subsets of a topological space \((X, \tau)\) is denoted by \( SO_{\lambda_{ac}}(X, \tau) \) or \( SO_{\lambda_{ac}}(X) \) (resp. \( SC_{\lambda_{ac}}(X, \tau) \) or \( SC_{\lambda_{ac}}(X) \)).

Proposition 2.12. For a topological space \( X \), \( SO_{\lambda_{ac}}(X) \subseteq SO_{\lambda}(X) \subseteq SO(X) \).

Proof. Obvious.

The following example shows that the converse of the above proposition may not be true in general.

Example 2.13. In Example 2.3, we have \( \{a, c\} \) is semi open but it is not \( \lambda^* \)-open. And also \( \{a, b\} \) is \( \lambda^* \)-open set but it is not \( \lambda_{ac} \)-open.

Definition 2.14. An \( s \)-operation \( \lambda \) on \( X \) is said to be \( s \)-regular which is equivalent to \( \lambda \)-regular [8] if for every semi open sets \( U \) and \( V \) of \( x \in X \), there exists a semi open set \( W \) containing \( x \) such that \( \lambda(W) \subseteq \lambda(U) \cap \lambda(V) \).

Definition 2.15. Let \( A \) be a subset of \( X \). Then:

(3) The \( \lambda_{ac} \)-closure of \( A \) (\( \lambda_{ac} Cl(A) \)) is the intersection of all \( \lambda_{ac} \)-closed sets containing \( A \).

(4) The \( \lambda_{ac} \)-interior of \( A \) (\( \lambda_{ac} Int(A) \)) is the union of all \( \lambda_{ac} \)-open sets of \( X \) contained in \( A \).

Proposition 2.16. For each point \( x \in X \), \( x \in \lambda_{ac} Cl(A) \) if and only if \( V \cap A \neq \phi \) for every \( V \in SO_{\lambda_{ac}}(X) \) such that \( x \in V \).

Proof. Obvious

Proposition 2.17. Let \( \{A_\alpha\}_{\alpha \in I} \) be any collection of \( \lambda_{ac} \)-open sets in a topological space \((X, \tau)\), then \( \bigcup_{\alpha \in I} A_\alpha \) is a \( \lambda_{ac} \)-open set.
Proof. Obvious

**Proposition 2.18.** Let $\lambda$ bean s-regular s-operation. If $A$ and $B$ are $\lambda_{ac}$-open sets in $X$, then $A \cap B$ is also a $\lambda_{ac}$-open set.

Proof. Obvious

3. Minimal $\lambda_{ac}$-Open Sets

**Definition 3.1.** Let $X$ be a space and $A \subseteq X$ be a $\lambda_{ac}$-open set. Then $A$ is called a minimal $\lambda_{ac}$-open set if $\phi$ and $A$ are the only $\lambda_{ac}$-open subsets of $A$.

**Example 3.2.** Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an s-operation $\lambda: SO(X) \rightarrow P(X)$ as $\lambda(A) = A$ if $A = \{a, c\}$ and $\lambda(A) = X$ otherwise. The $\lambda_{ac}$-open sets are $\phi, \{a, c\}$ and $X$. We have $\{a, c\}$ is minimal $\lambda_{ac}$-open set.

**Proposition 3.3.** Let $A$ be a nonempty $\lambda_{ac}$-open subset of a space $X$. If $A \subseteq \lambda_{ac} Cl(C)$, then $\lambda_{ac} Cl(A) = \lambda_{ac} Cl(C)$, for any nonempty subset $C$ of $A$.

**Proof.** For any nonempty subset $C$ of $A$, we have $\lambda_{ac} Cl(C) \subseteq \lambda_{ac} Cl(A)$. On the other hand, by supposition we see $\lambda_{ac} Cl(A) = \lambda_{ac} Cl(\lambda_{ac} Cl(C)) = \lambda_{ac} Cl(C)$ implies $\lambda_{ac} Cl(A) \subseteq \lambda_{ac} Cl(C)$. Therefore we have $\lambda_{ac} Cl(A) = \lambda_{ac} Cl(C)$ for any nonempty subset $C$ of $A$.

**Proposition 3.4.** Let $A$ be a nonempty $\lambda_{ac}$-open subset of a space $X$. If $\lambda_{ac} Cl(A) = \lambda_{ac} Cl(C)$, for any nonempty subset $C$ of $A$, then $A$ is a minimal $\lambda_{ac}$-open set.

**Proof.** Suppose that $A$ is not a minimal $\lambda_{ac}$-open set. Then there exists a nonempty $\lambda_{ac}$-open set $B$ such that $B \subseteq A$ and hence there exists an element $x \in A$ such that $x \notin B$. Then we have $\lambda_{ac} Cl(\{x\}) \subseteq X \setminus B$ implies that $\lambda_{ac} Cl(\{x\}) = \lambda_{ac} Cl(A)$. This contradiction proves the proposition.

**Remark 3.5.** In the remainder of this section we suppose that $\lambda$ is an s–regular operation defined on a topological space $X$.

**Proposition 3.6.** The following statements are true:

1. If $A$ is a minimal $\lambda_{ac}$-open set and $B$ a $\lambda_{ac}$-open set. Then $A \cap B = \phi$ or $A \subseteq B$.
2. If $B$ and $C$ are minimal $\lambda_{ac}$-open sets. Then $B \cap C = \phi$ or $B = C$.

**Proof.** (1) Let $B$ be a $\lambda_{ac}$-open set such that $A \cap B \neq \phi$. Since $A$ is a minimal $\lambda_{ac}$-open set and $A \cap B \subseteq A$, we have $A \cap B = A$. Therefore $A \subseteq B$.

2. If $A \cap B \neq \phi$, then by (1), we have $B \subseteq C$ and $C \subseteq B$. Therefore, $B = C$.

**Proposition 3.7.** Let $A$ be a minimal $\lambda_{ac}$-open set. If $x$ is an element of $A$, then $A \subseteq B$ for any $\lambda_{ac}$-open neighborhood $B$ of $x$.

**Proof.** Let $B$ be a $\lambda_{ac}$-open neighborhood of $x$ such that $A \notin B$. Since where $\lambda$ is $\lambda$–regular operation, then $A \cap B$ is $\lambda_{ac}$-open set such that $A \cap B \subseteq A$ and $A \cap B \neq \phi$. This contradicts our assumption that $A$ is a minimal $\lambda_{ac}$-open set.
Proposition 3.8. Let $A$ be a minimal $\lambda_{ac}$-open set. Then for any element $x$ of $A$, $A = \cap\{B:B$ is $\lambda_{ac}$-open neighborhood of $x\}$.

**Proof.** By Proposition 3.4, and the fact that $A$ is $\lambda_{ac}$-open neighborhood of $x$, we have $A \subseteq \cap\{B:B$ is $\lambda_{ac}$-open neighborhood of $x\} \subseteq A$. Therefore, the result follows.

Proposition 3.9. If $A$ is a minimal $\lambda_{ac}$-open set in $X$ not containing $x \in X$. Then for any $\lambda_{ac}$-open neighborhood $C$ of $x$, either $C \cap A = \phi$ or $A \subseteq C$.

**Proof.** Since $C$ is a $\lambda_{ac}$-open set, we have the result by Proposition 3.3.

Corollary 3.10. If $A$ is a minimal $\lambda_{ac}$-open set in $X$ not containing $x \in X$ such that $x \notin A$. If $\Lambda_x = \cap\{B:B$ is $\lambda_{ac}$-open neighborhood of $x\}$. Then either $\Lambda_x \cap A = \phi$ or $A \subseteq \Lambda_x$.

**Proof.** If $A \subseteq B$ for any $\lambda_{ac}$-open neighborhood $B$ of $x$, then $A \subseteq \cap\{B:B$ is $\lambda_{ac}$-open neighborhood of $x\}$. Therefore $A \subseteq \Lambda_x$. Otherwise there exists a $\lambda_{ac}$-open neighborhood $B$ of $x$ such that $B \cap A = \phi$. Then we have $\Lambda_x \cap A = \phi$.

Corollary 3.11. If $A$ is a nonempty minimal $\lambda_{ac}$-open set of $X$, then for a nonempty subset $C$ of $A$, $A \subseteq \lambda_{ac}Cl(C)$.

**Proof.** Let $C$ be any nonempty subset of $A$. Let $y \in A$ and $B$ be any $\lambda_{ac}$-open neighborhood of $y$. By Proposition 3.4, we have $A \subseteq B$ and $C = A \cap C \subseteq B \cap C$. Thus we have $B \cap C \neq \phi$ and hence $y \in \lambda_{ac}Cl(C)$. This implies that $A \cap \lambda_{ac}Cl(C)$. This completes the proof.

Combining Corollary 3.11 and Propositions 3.3 and 3.4, we have:

Theorem 3.11. Let $A$ be a nonempty $\lambda_{ac}$-open subset of space $X$. Then the following are equivalent:

1. $A$ is minimal $\lambda_{ac}$-open set, where $\lambda$ is $s$-regular.
2. For any nonempty subset $C$ of $A$, $A \subseteq \lambda_{ac}Cl(C)$.
3. For any nonempty subset $C$ of $A$, $\lambda_{ac}Cl(A) = \lambda_{ac}Cl(C)$.

4. Finite $\lambda_{ac}$-Open Sets

In this section, we study some properties of minimal $\lambda_{ac}$-open sets in finite $\lambda_{ac}$-open sets and $\lambda_{ac}$-locally finite spaces.

Proposition 4.1. Let $(X, \tau)$ be a topological space and $\phi \neq B$ a finite $\lambda_{ac}$-open set in $X$. Then there exists at least one (finite) minimal $\lambda_{ac}$-open set $A$ such that $A \subseteq B$.

**Proof.** Suppose that $B$ is a finite $\lambda_{ac}$-open set in $X$. Then we have the following two possibilities:

1. $B$ is a minimal $\lambda_{ac}$-open set.
2. $B$ is not a minimal $b$-open set.
In case (1), if we choose \( B = A \), then the proposition is proved. If the case (2) is true, then there exists a nonempty (finite) \( \lambda_{ac} \)-open set \( B_1 \) which is properly contained in \( B \). If \( B_1 \) is minimal \( \lambda_{ac} \)-open, we take \( A = B_1 \). If \( B_1 \) is not a minimal \( \lambda_{ac} \)-open set, then there exists a nonempty (finite) \( \lambda_{ac} \)-open set \( B_2 \) such that \( B_2 \subseteq B_1 \subseteq B \). We continue this process and have a sequence of \( \lambda_{ac} \)-open sets... \( \subseteq B_m \subseteq \cdots \subseteq B_2 \subseteq B_1 \subseteq B \). Since \( B \) is a finite, this process will end in a finite number of steps. That is, for some natural number \( k \), we have a minimal \( \lambda_{ac} \)-open set \( B_k \) such that \( B_k = A \). This completes the proof.

**Definition 4.2.** A space \( X \) is said to be a \( \lambda_{ac} \)-locally finite space, if for each \( x \in X \) there exists a finite \( \lambda_{ac} \)-open set \( A \) in \( X \) such that \( x \in A \).

**Corollary 4.3.** Let \( X \) be a \( \lambda_{ac} \)-locally finite space and \( B \) a nonempty \( \lambda_{ac} \)-open set. Then there exists at least one (finite) minimal \( \lambda_{ac} \)-open set \( A \) such that \( A \subseteq B \), where \( \lambda \) is semi-regular.

**Proof.** Since \( B \) is a nonempty set, there exists an element \( x \) of \( B \). Since \( X \) is a \( \lambda_{ac} \)-locally finite space, we have a finite \( \lambda_{ac} \)-open set \( B_x \) such that \( x \in B_x \). Since \( B \cap B_x \) is a finite \( \lambda_{ac} \)-open set, we get a minimal \( \lambda_{ac} \)-open set \( A \) such that \( A \subseteq B \cap B_x \subseteq B \) by Proposition 4.1.

**Proposition 4.4.** Let \( X \) be a space and for any \( \alpha \in I, B_\alpha \) a \( \lambda_{ac} \)-open set and \( \phi \neq A \) a finite \( \lambda_{ac} \)-open set. Then \( A \cap (\bigcap_{\alpha \in I} B_\alpha) \) is a finite \( \lambda_{ac} \)-open set, where \( \lambda \) is semi-regular.

**Proof.** We see that there exists an integer \( n \) such that \( A \cap (\bigcap_{\alpha \in I} B_\alpha) = A \cap (\bigcap_{i=1}^{n} B_{\alpha_i}) \) and hence we have the result.

Using Proposition 4.4, we can prove the following:

**Theorem 4.5.** Let \( X \) be a space and for any \( \alpha \in I, B_\alpha \) a \( \lambda_{ac} \)-open set and for any \( \beta \in J, B_\beta \) a nonempty finite \( \lambda_{ac} \)-open set. Then \( (\bigcup_{\beta \in J} B_\beta) \cap (\bigcap_{\alpha \in I} B_\alpha) \) is a \( \lambda_{ac} \)-open set, where \( \lambda \) is semi-regular.

5. More Properties

Let \( A \) be a nonempty finite \( \lambda_{ac} \)-open set. It is clear, by Proposition 3.3 and Proposition 4.1, that if \( \lambda \) is semi-regular, then there exists a natural number \( m \) such that \( \{A_1, A_2, \ldots, A_m\} \) is the class of all minimal \( \lambda_{ac} \)-open sets in \( A \) satisfying the following two conditions:

1. For any \( i, n \) with \( 1 \leq i \), \( n \leq m \) and \( i \neq n, A_i \cap A_n = \phi \).
2. If \( C \) is a minimal \( \lambda_{ac} \)-open set in \( A \), then there exists \( i \) with \( 1 \leq i \leq m \) such that \( C = A_i \).
Theorem 5.1. Let $X$ be a space and $\phi \neq A$ a finite $\lambda_{ac}$-open set such that $A$ is not a minimal $\lambda_{ac}$-open set. Let $\{A_1, A_2, ..., A_m\}$ be a class of all minimal $\lambda_{ac}$-open sets in $A$ and $y \in A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$. Define $A_y = \bigcap \{B: B$ is $\lambda_{ac}$-open neighborhood of $x \}$. Then there exists a natural number $k \in \{1, 2, 3, ..., m\}$ such that $A_k$ is contained in $A_y$, where $\lambda$ is semi-regular.

Proof. Suppose on the contrary that for any natural number $k \in \{1, 2, 3, ..., m\}$, $A_k$ is not contained in $A_y$. By Corollary 3.7, for any minimal $\lambda_{ac}$-open set $A_k$ in $A$, $A_k \cap A_y = \phi$. By Proposition 4.4, $\phi \neq A_y$ is a finite $\lambda_{ac}$-open set. Therefore by Proposition 4.1, there exists a minimal $\lambda_{ac}$-open set $C$ such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, we have $C$ is a minimal $\lambda_{ac}$-open set in $A$. By supposition, for any minimal $\lambda_{ac}$-open set $A_k$, we have $A_k \cap C \subseteq A_k \cap A_y = \phi$. Therefore, for any natural number $k \in \{1, 2, 3, ..., m\}, C \neq A_k$. This contradicts our assumption. Hence the proof.

Proposition 5.2. Let $X$ be a space and $\phi \neq A$ be a finite $\lambda_{ac}$-open set which is not a minimal $\lambda_{ac}$-open set. Let $\{A_1, A_2, ..., A_m\}$ be a class of all minimal $\lambda_{ac}$-open sets in $A$ and $y \in A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, 3, ..., m\}$, such that for any $\lambda_{ac}$-open neighborhood $B_y$ of $y$, $A_k$ is contained in $B_y$, where $\lambda$ is $\lambda$-regular.

Proof. This follows from Theorem 5.1, as $\bigcap \{B: B$ is $\lambda_{ac}$-open of $y \} \subseteq B_y$. Hence the proof.

Theorem 5.3. Let $X$ be a space and $\phi \neq A$ be a finite $\lambda_{ac}$-open set which is not a minimal $\lambda_{ac}$-open set. Let $\{A_1, A_2, ..., A_m\}$ be the class of all minimal $\lambda_{ac}$-open sets in $A$ and $y \in A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, 3, ..., m\}$, such that $y \in \lambda_{ac} Cl(A_k)$. where $\lambda$ is $\lambda$-regular.

Proof. It follows from Proposition 5.2, that there exists a natural number $k \in \{1, 2, 3, ..., m\}$ such that $A_k \subseteq B$ for any $\lambda_{ac}$-open neighborhood $B$ of $y$. Therefore $\phi \neq A_k \cap A_k \subseteq A_k \cap B$ implies $y \in \lambda_{ac} Cl(A_k)$. This completes the proof.

Proposition 5.4. Let $\phi \neq A$ be a finite $\lambda_{ac}$-open set in a space $X$ and for each $k \in \{1, 2, 3, ..., m\}, A_k$ is a minimal $\lambda_{ac}$-open sets in $A$. If the class $\{A_1, A_2, ..., A_m\}$ contains all minimal $\lambda_{ac}$-open sets in $A$, then for any $\phi \neq B_k \subseteq A_k, A \subseteq \lambda_{ac} Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$, where $\lambda$ is semi-regular.

Proof. If $A$ is a minimal $\lambda_{ac}$-open set, then this is the result of Theorem 3.11 (2). Otherwise, when $A$ is not a minimal $\lambda_{ac}$-open set. If $x$ is any element of $A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$, then by Theorem 5.3, $x \in \lambda_{ac} Cl(A_1) \cup \lambda_{ac} Cl(A_2) \cup ... \cup \lambda_{ac} Cl(A_m)$. Therefore, by Theorem 3.11 (3), we obtain that $A \subseteq \lambda_{ac} Cl(A_1) \cup$
\[ \lambda_{ac} \text{Cl}(A_2) \cup ... \cup \lambda_{ac} \text{Cl}(A_m) = \lambda_{ac} \text{Cl}(B_1) \cup \lambda_{ac} \text{Cl}(B_2) \cup ... \cup \lambda_{ac} \text{Cl}(B_m) = \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m). \]

**Proposition 5.5.** Let \( \phi \neq A \) be a finite \( \lambda_{ac} \)-open set and \( A_k \) is a minimal \( \lambda_{ac} \)-open set in \( A \), for each \( k \in \{1,2,3,...,m\} \). If for any \( \phi \neq B_k \subseteq A_k, A \subseteq \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \) then \( \lambda_{ac} \text{Cl}(A) = \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m). \)

**Proof.** For any \( \phi \neq B_k \subseteq A_k \) with \( k \in \{1,2,3,...,m\} \), we have \( \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \subseteq \lambda_{ac} \text{Cl}(A) \). Also, we have \( \lambda_{ac} \text{Cl}(A) \subseteq \lambda_{ac} \text{Cl}(B_1) \cup \lambda_{ac} \text{Cl}(B_2) \cup ... \cup \lambda_{ac} \text{Cl}(B_m) = \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \). Therefore, \( \lambda_{ac} \text{Cl}(A) = \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \) for any nonempty subset \( B_k \) of \( A_k \) with \( k \in \{1,2,3,...,m\} \).

**Proposition 5.6.** Let \( \phi \neq A \) be a finite \( \lambda_{ac} \)-open set and for each \( k \in \{1,2,3,...,m\} \), \( A_k \) is a minimal \( \lambda_{ac} \)-open set in \( A \). If for any \( \phi \neq B_k \subseteq A_k, \lambda_{ac} \text{Cl}(A) = \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \), then the class \( \{A_1, A_2,...,A_m\} \) contains all minimal \( \lambda_{ac} \)-open sets in \( A \).

**Proof.** Suppose that \( C \) is a minimal \( \lambda_{ac} \)-open set in \( A \) and \( C \neq A_k \) for \( k \in \{1,2,3,...,m\} \). Then we have \( C \cap \lambda_{ac} \text{Cl}(A_k) = \phi \) for each \( k \in \{1,2,3,...,m\} \). It follows that any element of \( C \) is not contained in \( \lambda_{ac} \text{Cl}(A_1 \cup A_2 \cup ... \cup A_m) \). This is a contradiction to the fact that \( C \subseteq A \subseteq \lambda_{ac} \text{Cl}(A) = \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \). This completes the proof.

Combining Propositions 5.4, 5.5 and 5.6, we have the following theorem:

**Theorem 5.7.** Let \( A \) be a nonempty finite \( \lambda_{ac} \)-open set and \( A_k \) a minimal \( \lambda_{ac} \)-open set in \( A \) for each \( k \in \{1,2,3,...,m\} \). Then the following three conditions are equivalent:

1. The class \( \{A_1, A_2,...,A_m\} \) contains all minimal \( \lambda_{ac} \)-open sets in \( A \).
2. For any \( \phi \neq B_k \subseteq A_k, A \subseteq \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \).
3. For any \( \phi \neq B_k \subseteq A_k, \lambda_{ac} \text{Cl}(A) = \lambda_{ac} \text{Cl}(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \), where \( \lambda \) is semi-regular.

**References**


