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Maximal (k, n)-arc in Projective Plane PG(2, 5)

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Abstract:

In this paper we recognize maximal (k, n)-arcs in the projective plane PG (2, 5), n = 2, 3, ...,5, where a (k, n)-arc K in a projective plane is a set of K points such that no n + 1 of which are collinear. A (k, n) – arc is a maximal if and only if every line in the projective plane PG (2, P) is a O-secant, or n-secant, which represented as (k, 2)-arc and (k, 6)-arc. A (k, n)-arc is complete if it is not contained in a (k + 1, n) – arc.

Keywords:

projective plane PG (2, 5), conics of PG (2, 5), maximal and complete (k, n)-arcs.

1. Introduction:

Ahmad(1999) [4] studied the complete arcs in the projective plane over Galois Field GF(9), also Rashad (1999) [10] showed the complete arcs in the projective plane over Galois Field GF(q), and Massa (2004) [7] studied the constriction of (k, n)- arcs from (k, m) – arcs in the PG(2, 17) for $2 \leq m < n$. Ban (2001) [5] studied maximal (k, n)-arcs. Finally Najim (2005) [8] studied the constriction of (k, n)- arcs from (k, m) – arcs in the PG(2, 13) for $2 \leq m < n$. This paper deals with maximal (k, n)– arc in the projective plane PG (2, n) which are three (6, 2)-arcs and unique (31, 6)-arc.

The maximal (k, n)-arcs are of two types which are (k, 2) – arcs where each line contains six points and (k,6) – arc or which represented the whole plane where each line contains eight points. From the both maximal (k, n)-arcs we construct complete (k , n) – arc , n $<$ k prepared from the intersecting of some maximal or complete (k, m) – arc, $2 \leq m < n$, after eliminating some points of incomplete the new constructing arcs .

2- Basic Definition:

2.1 Definition (K, n) – Arcs [1, 2, 6, 8] :

A (k, n) – arc in the projective plane PG (2, P) is a set K points such that some line meets K in n points but no line meets K in more than n points, $n \geq 2$, p is prime .

2.2 Definition [4,7, 9, 10] :

A (k, n) –arc is complete if it is not contained in (k + 1, n) – arc.

2.3 Definition [3, 6, 8, 11] :

A point P which is not on (k, n) – arc K has index i if there are exactly i (n – secant) through P, we dented the numbers of point P of index i by C_i

2.4 Definition [5, 6, 9, 11] :

A (k, n) – arc K is a maximal if and only if every line in PG (2, p) is a 0 – secant or n – secant.

2.5 Definition PG(2, 5)[1, 6, 9, 11]:

A PG (2, 5) is the two – dimensional projective space which consists of points and lines with incidence relation between them and satisfying the following axioms:

i. Any two distinct lines are intersected in a unique point.

ii. Any two distinct points are contained in a unique line.

iii. There exist at least four points such that no three of them are collinear .

Remark (1)[4, 5, 6] :

A (k, n)–arc K is complete if and only if $C_0 = 0$,we mean that C_0 is 0 (n – secant), thus K is complete if and only if every point of PG (2, p) lies on some (n–secant) of K.

3- The Projective Plane PG (2, 5):

The projective plane PG (2, 5) contains 31 points and 31 lines, every line contains 6 points and every point is on 6 lines. Any line in PG (2, 5) can be constructed by means of variety v. Let Pi and Li , i = 1 , 2 , …,31 be the points and lines of PG (2, 5) respectively. Let i stands for the points Pi and the lines Li, then all the points and the lines in PG (2,5) are given in the table (1)

I	Pi	Li					
1	(1, 0, 0)	2	7	12	17	22	27
2	(0, 1, 0)	1	7	8	9	10	11
3	(1, 1, 0)	6	7	16	20	24	28
4	(2, 1, 0)	4	7	14	21	23	30
5	(3, 1, 0)	5	7	15	18	26	29
6	(4, 1, 0)	3	7	13	19	25	31
7	(0, 0, 1)	1	2	3	4	5	6
8	(1,0,1)	2	11	16	21	26	31
9	(2, 0, 1)	2	9	14	19	24	29
10	(3, 0, 1)	2	10	15	20	25	30
11	(4, 0, 1)	2	8	13	18	23	28
12	(0, 1, 1)	1	27	28	29	30	31
13	(1, 1, 1)	6	11	15	19	23	27
14	(2, 1, 1)	4	9	16	18	25	27
15	(3, 1 , 1)	5	10	13	21	24	27
16	(4, 1 , 1)	3	8	14	20	26	27
17	(0, 2, 1)	1	17	18	19	20	21
18	(1, 2, 1)	5	11	14	17	25	28
19	(2, 2, 1)	6	9	13	17	26	30
20	(3, 2, 1)	3	10	16	17	23	29
21	(4 , 2, 1)	4	8	15	17	24	31
22	(0 , 3, 1)	1	22	23	24	25	26

23	(1 , 3, 1)	4	11	13	20	22	29
24	(2, 3, 1)	3	9	15	21	22	28
25	(3, 3 ,1)	6	10	14	18	22	31
26	(4, 3, 1)	5	8	16	19	22	30
27	(0, 4, 1)	1	12	13	14	15	16
28	(1, 4, 1)	3	11	12	18	24	30
29	(2, 4, 1)	5	9	12	20	23	31
30	(3, 4, 1)	4	10	12	19	26	28
31	(4, 4, 1)	6	8	12	21	25	29

Table (1)

(Contains 31 points and 31 lines, every line contains 6 points and every point is on 6 lines)

4- The Construction of (k, 2) – Arcs in PG (2, 5) :

Let A = {1,2,7,13} be the set reference and unit points in the table (1) such that 1 = (1, 0, 0), 2 = (0, 1, 0), 7 = (0 , 0, 1), 13= (1, 1, 1) . A is a (4, 2) – arc , since no three points of A are collinear , the points of A are the vertices of a quadrangle whose side are the lines.

ℓ_1 = [1,2] = { 1,2,3,4,5,6 }

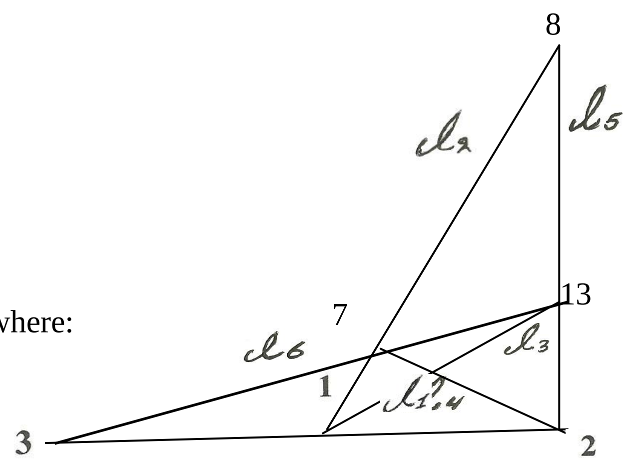
ℓ_2 = [1,7] = { 1,7,8,9,10,11 }

ℓ_3 = [1,13] = { 1,12,13,14,15,16 }

ℓ_4 = [2, 7] = {2, 7, 12, 17, 22, 27}

ℓ_5 = [2, 13] = {2, 8, 13, 18, 23, 28}

ℓ_6 = [7, 13] = {3, 7, 13, 19, 25, 31}



The diagonal points of A are the points {3, 8, 12} where:

3 = $\ell_1 \cap \ell_6$

8 = $\ell_2 \cap \ell_5$

12 = $\ell_3 \cap \ell_4$, which are the intersection points of pairs of the opposite sides. Then there are 25 points on the sides of the quadrangle , four of them are points of the arc A , and three of

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them are diagonal points of A , so there are six points not on the sides of the quadrangle which are the points of index zero for A these points are: {20, 21, 24, 26, 29, 30}. Hence A is incomplete (4, 2) – arc.

5- The Conics In PG (2, 5) Through the Reference and Unit points

The general equation of conic is

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0 \qquad (1)$$

By substituting the points of the arc – A in (1), we get

1 = (1, 0, 0) → $a_1 = 0$

2 = (0, 1, 0) → $a_2 = 0$

7 = (0, 0, 1) → $a_3 = 0$

13 = (1, 1, 1) → $a_4 + a_5 + a_6 = 0$

So equation (1) becomes

$$a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0 \qquad (2)$$

If $a_4 = 0$, then $a_5 x_1 x_3 + a_6 x_2 x_3 = 0$

Hence $x_3 (a_5 x_1 + a_6 x_2) = 0$, $x_3 = 0$ or $a_5 x_1 + a_6 x_2 = 0$

which are a pair of lines, then the conic is degenerated, therefore $a_4 \neq 0$

Similarly $a_5 \neq 0$ *and* $a_6 \neq 0$

Dividing equation (2) by a_4 we get

$$x_1 x_2 + \frac{a_5}{a_4} x_1 x_3 + \frac{a_6}{a_4} x_2 x_3 = 0$$

$$x_1 x_2 + \alpha x_1 x_3 + \beta x_2 x_3 = 0 \qquad (3)$$

$\alpha = \frac{a_5}{a_4}$, $\beta = \frac{a_6}{a_4}$, then

$1 + \alpha + \beta = 0$ (mod (5))

$\beta = -(1 + \alpha)$, then (3) can be written as:

$$x_1 x_2 + \alpha x_1 x_3 - (1 + \alpha) x_2 x_3 = 0 \qquad (4)$$

Where $\alpha \neq 0$ and $\alpha \neq 4$, for if $\alpha = 0$ or $\alpha = 4$, we get degenerated conic, i.e $\alpha = 1, 2, 3$.

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6-The Equations and the Points of the Conic of PG (2, 5) Through The Reference and Unit Points:-

For any value for α there is a unique conic containing six points, four of them are the reference and unit points and they are maximal arcs since contains six points:

1- If $\alpha = 1$, then the equation of the conic C_1 is

$x_1 x_2 + x_1 x_3 + 3 x_2 x_3 = 0$, the point of C_1 are {1, 2, 7, 13, 14, 20},

2- If $\alpha = 2$, then the equation of the conic C_2 is

$x_1 x_2 + 2 x_1 x_3 + 2x_2 x_3 = 0$, the points of C_2 are {1, 2, 7, 13, 21, 23} ,

3- If $\alpha = 3$, then the equation of the conic C_3 is

$x_1 x_2 + 3x_1 x_3 + x_2 x_3 = 0$, the points of C_3 are {1, 2, 7, 13, 24, 30}.

Thus there are three maximal (6, 2) – arcs in the PG (2, 5) which are

C_1 = {1, 2, 7,1 3, 14, 20}

C_2 = {1, 2, 7, 13, 21, 23}

C_3 = {1, 2, 7, 13, 24, 30}

each of these arcs has no points of index zero so they are complete.

Now we know that (K, 6)-arc in PG (2, 5) which represented as whole plane is also maximal arc. So we can construct the complete (K, n)-arc from the whole plane and some maximal (6, 2)-arcs as follows:-

6.1-Construction of Complete (k, 5)–Arcs in PG(2, 5) from Maximal Arcs

The complete (k, 5) – arcs can be constructed by eliminating some of maximal (K, 2) - arc as follows:

From the maximal whole plane W={ 1, 2, 3,…, 31}, and C_1

let $H_1 = W-C_{1=}$ {3,4,5,6,8,9,10,11,12,15,16,17,18,19,21,22,23,24,25,26,27,28,29,30,31}, we notice that there are some line meet H_1 in six points , hence(K, 5) is note complete ,so we eliminate some point from H_1 to determine a complete (K, 5)-arc as follows :

$H_1^* = (W- C_1)$ /{4, 23}=

{3,5,6,8,9,10,11,12,15,16,17,18,19,21,22,24,25,26,27,28,29,30,31}.

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We notice that H_1^* is incomplete (k, 5) – arc, since the point {20} is of index zero, therfore we add this point to H_1^*, H_1^* become complete (K, 5) - arc.

Let H_2 = W- C_2 ={3,4,5,6,8,9,10,11,12,14,15,16,17,18,19,20,22,24,25,26,27,28,29,30,31}, we notice that there are some line meet H_2 in six point hence H_2 is not complete .So we eliminate some points from H_2 to determine a complete (K, 5) - arc as follows :

Let H_2^* =(W- C_2) /{12,17,27}={3,4,5,6,8,9,10,11,14,15,16,18,19,20, ,22,24,25,26, 28,29,30,31} ,and H_2^* is a complete (K, 5)-arc since the set of index zero=0 .

Let H_3= W-C_3={3,4,5,6,8,9,10,11,12,14,15,16,17,18,19,20,21,22,23,25,26,27,28,29,31}, we notice that there some lines meet H_3 six points ,hence H_3 is not complete .So we eliminate some points from H_3 to determine a complete(K, 5)-arc, as follows:

Let H_3^*=(W-C_3) /{10,12,27,28}={3,4,5,6,8,9,11,14,15,16,17,18,19,20,21,22,23,25,26,29,31} and H_3^* is incomplete since the point {1} is of index zero , and we add this point to H_3^*, H_3^*={1,3,4,5,6,8,9,11,14,15,16,17,18,19,20,21,22,23,25,26,29,31},then H_3^* is complete(22, 5) -arc.

6.2 Construction of Complete (k, 4) – Arcs in PG (2, 5):

The complete (K, 4)-arc constructed by intersecting two complete (K, 5) arcs as follows:

I_1= $H_1^* \cap H_2^*$={3,5,6,10,11,15,16,18,19,20,22 ,24,25,26,27,28,29,30,31} is incomplete (K,4)-arc since there are some line meet the arc in five points so we eliminate some points from the arc to determine a complete (K, 4) - arc as follows:

I_1^*= $H_1^* \cap H_2^*$-{5,6,30}={3,10,11,15,16,18,19,20,22,24,25,26,27,28,29,31},we notice that this arc is a complete (16, 4)-arc .

I_2= $H_1^* \cap H_3^*$={3,5,6,8,9,11,15,16,17,18,19,21,22,25,26,29,31} is incomplete (K,4)-arc since there are some line meet I_2 in five points , so we eliminate some points from it to determine a complete (K, 4) - arc as follows

I_2^*= $H_1^* \cap H_3^*$-{5,6,11,22}={3,8,9,15,16,17,18,19,21,25,26,29,31} we notice that this arc incomplete since $C_0 \neq 0$, C_0 ={14} ,and we add {14} to I_2^* to be complete I_2^*={3,8,9,14,15,16,17,18,19,21,25,26,29,31}.

I_3= $H_2^* \cap H_3^*$={3,4,5,6,8,9,11,14,15,16,18,19,20,22,25,26,29,31}it is incomplete (K, 4) –arc since there are some line meet I_3 in five points so we eliminate some points from it to determine a complete (K, 4)-arc as follows :

$I_3^* = H_2^* \cap H_3^*$-{4,5,6,8}={3,9,11,14,15,16,18,19,20,22,25,26,29,31} we notice that this arc is incomplete since $C_0 \neq 0$,C_0={1,12,17,23,30} ,and we add {1,30}to I_3^* ,I_3^* become complete ,
I_3^* ={1,3,9,11,14,15,16,18,19,20,22,25,26,29,30,31}

6.3- Construction of Complete (K, 3) – Arcs in PG(2, 5) :-

The complete (K, 3) –arc constructed by intersecting two complete (k, 4) - arcs as follows:
$J_1 = I_1^* \cap I_2^*$ ={3,15,16,18,19,25,26,29,31} , is incomplete (K, 3) –arc since there are some lines meet this arc in four points so we eliminate some points from it to determine a complete (K, 3)-are as follows :
$J_1^* = I_1^* \cap I_2^*$ -{3,15}={16,18,19,25,26,29,31} , we notice that this arc is incomplete (K, 3)-arc since $C_0 \neq 0$,C_0={6,8,10,12,14,17,20}.
We add {6, 17} to J_1^*, J_1^* become complete
J_1^*={6,16,17,18,19,25,26,29 ,31}
$J_2 = I_1^* \cap I_3^*$={3,11,15,16,18,19,20,22,25,26,29,31} is incomplete (K, 3) – arc ,since there are some line meet this arc in four points , so we eliminate some points from it to determine a complete (K, 3) – arc as follows :
$J_2^* = I_1^* \cap I_3^*$–{16,18,19}={3,11,15,20,22,25,26,29,31} ,we notice that this arc is incomplete(K, 3) –arc since $C_0 \neq 0$, C_0={6,12,17}. We add{6,17} to J_3^* , J_3^* become complete ,
J_2^*={3,6,11,15,17,20,22,25,26,29,31} is a complete (11,3) – arc .
$J_3 = I_2^* \cap I_3^*$={3,9,14,15,16,18,19,25,26,29,31} is incomplete (K, 3) arc since there are some line meet this arc in four points so we eliminate some points from it to determine a complete(k, 3) -arc as follows :
$J_3^* = I_2^* \cap I_3^*$-{18,19}={3,9,14,15,16,25,26,29,31}, we notice that the arc is incomplete since $C_0 \neq 0$, C_0={6,30},and we add {6} to J_3^*, J_3^* become complete (10, 3) - arc. J_3= {3,6,9,14,15,16, 25, 26, 29, 31}.

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