http://garmian.edu.krd https://doi.org/10.24271/garmian.128

On $Rad-$ Supplemented Modules, Weak $Rad-$ Supplemented Modules and Completely Weak $Rad-$ Supplemented Modules

Adil Kadir Jabbar[1] and Bakhtyar Mahmood Rahim[2]
[1]Department of Mathematics, College of Science, University of Sulaimani, Sulaimani, Iraq
[2]Department of Mathematics, College of Education, University of Sulaimani, Sulaimani, Iraq
[1]adilkj@gmail.com or adil.jabbar@univsul.edu.iq [2]bakhtyar1987@gmail.com

Abstract

In this paper some results that concerning localization of modules are proved. It also, studies the effect of localization on certain types of modules such as $Rad-$supplemented modules, weak $Rad-$supplemented modules and completely weak $Rad-$supplemented modules. Several conditions are given under which certain properties of such types of algebraic structures are preserved under localization.

Keywords: Localization of modules, $Rad-$supplemented modules, weak $Rad-$supplemented modules, completely weak $Rad-$supplemented modules and amply $Rad-$supplemented modules.

1 Introduction

Throughout this paper, R is a commutative ring with identity and M is a unitary left $R-$module. By $N \leq M$ we mean N is a submodule of M. A submodule V of M is called a small submodule of M, denoted by $V \ll M$, if $L \leq M$ is any submodule such that $V + L = M$, then $L = M$ **[12]**, and V is called a supplement (a weak supplement) of $U \leq M$ if $M = U + V$ and $U \cap V \ll V$ ($U \cap V \ll M$) **[8]**. Moreover, M is called a supplemented (a weak supplemented) module if every submodule of M has a supplement (a weak supplement) in M **[4]**. M is called an amply supplemented $R-$module if for any submodules U and V of M with $M = U + V$ there exists a submodule K of U such that $K \leq V$ **[4]**, and M is called an amply weak supplemented $R-$ module if every submodule of M has amply supplement in M **[8]** and V is called a $Rad-$supplement or a generalized supplement (a weak $Rad-$supplement or a generalized weak supplement) of U in M if $M = U + V$ and $U \cap V \leq RadV$ ($U \cap V \leq RadM$) **[9]**. Moreover, M is called a $Rad-$ supplemented or a generalized supplemented (a weakly $Rad-$supplemented or a generalized weakly supplemented) module if every submodule of M has a $Rad-$supplement or a generalized supplement (a weak $Rad-$supplement or a generalized weak supplement) in M **[11]**. M is called completely weak $Rad-$ supplemented if every submodule of M is weakly $Rad-$supplemented **[9]** and it is called amply $Rad-$supplemented (or generalized amply supplemented) in case $M = U + V$ implies that U has a $Rad-$supplement (or has a generalized supplement) in V **[13]**. If $r \in R,$ then we define $N:r = \{x \in M\ ; rx \in N\}$ **[1]**. A submodule $K \leq M$ is called a radical submodule if $Rad(K) = K$. M is called a hollow module if every proper submodule of M is small in M **[13]**, and by a hollow radical submodule is meant a submodule which is both hollow and radical. M is called a semi-simple if every submodule of M is a direct summand **[5]**. For a submodule K of M we define $S_M(K) = \{r \in R: rx \in K,$ for some $x \notin K\}$ **[2]**. A non-empty subset S of R is called a multiplicative system in R, if $0 \notin S$ and $a, b \in S$ implies $ab \in S$ **[6]**. If S is a multiplicative system in R, then one can obtain an R_S −module, denoted by M_S, under the module operations $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $\frac{r}{u} . \frac{x}{s} = \frac{rx}{us}$, for $\frac{r}{u} \in R_S$ and $\frac{x}{s}, \frac{y}{t} \in M_S$, so that when we say M_S is a module we mean M_S is an R_S −module. In fact, this module M_S is known as the localization of M at the multiplicative system S **[1]**.

2. Some Basic Preliminaries

The following are some known results on which we depend to prove the main results of this paper.

Corollary 2.1. **[3]** Let M be an $R-$module and P a prime ideal of R. For submodules N, L of M the following are satisfied.

(1) $(M/N)_P \cong M_P/N_P$.
(2) $(N + L)_P \cong N_P + L_P$.
(3) $(N \cap L)_P \cong N_P \cap L_P$.

Proposition 2.2. **[7]** Let L and N be submodules of an $R-$module M. Then $L \subseteq N$ if and only if $L_P \subseteq N_P$ for every maximal ideal P of R.

Corollary 2.3. **[7]** Let L and N be submodules of an $R-$module M. Then
$L = N$ if and only if $L_P = N_P$, for every maximal ideal P of R.
In particular, $N = M$ if and only if $N_P = M_P$, for every maximal ideal P of R.

Lemma 2.4. **[11]** Let M be an R −module and V a Rad −supplement submodule of U in M. If $U \cap V$ is a supplement submodule in U, then V is supplement submodule in M.

3. Main Results

First, we give an example of an R −module M in which there is a prime ideal P of R and for which $S(K) >\subseteq P$, for all proper submodules K of M.

Example. Consider Z_8 as a Z −module, that is take $R = Z$ and $M = Z_8$. Clearly $P = <2>$ is a prime ideal of Z. The only proper submodules of Z_8 are $\{0\}, \{0,4\}, \{0,2,4,6\}$. Now, one can easily calculate $S(\{0\}), S(\{0,4\})$ and $S(\{0,2,4,6\})$ and get that
$S(\{0\}) = <2> = P \subseteq P$.

$S(\{0,4\}) =< 4 >\subseteq P$.
$S(\{0,2,4,6\}) =< 8 >\subseteq P$.
Hence, Z_8 is a Z −module, where $P =< 2 >$ is a prime ideal of Z and such that $S(K) \subseteq P$ for all submodules K of Z_8.

Now, we prove the first result.

Lemma 3.1. Let M be an R −module with submodules U and V of M and P be any maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If V_P is a weak Rad −supplement submodule of U_P in M_P, then for a submodule L' of U_P , there exists a submodule L of U such that $\frac{V+L}{L}$ is a weak Rad −supplement submodule of $\frac{U}{L}$ in $\frac{M}{L}$.

Proof. As V_P is a weak Rad −supplement submodule of U_P in M_P, we have $U_P + V_P = M_P$ and $U_P \cap V_P \leq Rad(M_P)$, this implies that $(U+V)_P = M_P$ and $(U \cap V)_P \leq Rad(M_P)$. By using **[1, Corollary 3.26]**, we have $(U \cap V)_P \leq (RadM)_P$, so by **Corollary 2.3** and **Proposition 2.2**, we get $U + V = M$ and $U \cap V \leq RadM$, so that V is a weak Rad −supplement submodule of U in M. Now, since $L' \leq U_P$, so by **[1, Lemma 3.16]**, there exists a submodule $L \leq U$ such that $L' = L_P$. Thus by **[9, Lemma 2.1]**, we get $\frac{V+L}{L}$ is a weak Rad −supplement submodule of $\frac{U}{L}$ in $\frac{M}{L}$.

In **[9, Proposition II.1]**, it is proved that every factor module of completely weak Rad −supplemented module is also a completely weak Rad −supplemented module. Now, we prove this result by replacing the module with its localization at maximal ideals.

Proposition 3.2. Let M be an R −module with a submodule L of M and P be any maximal ideal of R such that for each proper submodule K of M we have $S(K) \subseteq P$. If M_P is a completely weak Rad −supplemented R_P −module, then $\frac{M}{L}$ is a completely weak Rad −supplemented R −module.

Proof. Let $\frac{K}{L}$ be a submodule of $\frac{M}{L}$, where $L \leq K \leq M$. Then, $\frac{K_P}{L_P}$ is a submodule of $\frac{M_P}{L_P}$, where $L_P \leq K_P \leq M_P$. Let $\frac{U_P}{L_P}$ be a submodule of $\frac{K_P}{L_P}$, where $L_P \leq U_P \leq K_P$. Since M_P is a completely weak Rad −supplemented R_P −module, there exists a submodule V_P of K_P for which $U_P + V_P = K_P$ and $U_P \cap V_P \leq Rad(K_P)$. As V_P is a weak Rad −supplement of U_P in K_P and $L_P \leq U_P$, so by **Lemma 3.1**, we get that $\frac{V+L}{L}$ is a weak Rad − supplement of $\frac{U}{L}$ in $\frac{K}{L}$. Hence $\frac{K}{L}$ is a weakly Rad −supplemented module. So that, $\frac{M}{L}$ is a completely weak Rad −supplemented R −module.

In the next result, we prove that, if the localization of a module at a maximal ideal is completely weak Rad −supplemented, then the module itself is so.

Proposition 3.3. Let M be an $R-$module and P be any prime ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If M_P is a completely weak $Rad-$ supplemented R_P- module, then M is also a completely weak $Rad-$supplemented $R-$module.

Proof. Let W be any submodule of $R-$module M and K be any submodule of W. Then, by P**roposition 2.2**, W_P is a submodule of M_P and K_P is a submodule of W_P. Since M_P is completely weak $Rad-$ supplemented, then, W_P is weak $Rad-$supplemented, therefore, K_P has a weak $Rad-$supplemented in M_P. This implies that, there exists a submodule U' of W_P such that $U' + K_P = W_P$ and $U' \cap K_P \leq Rad(W_P)$. By **[1, Lemma 3.16]**, there exists a submodule U of W such that $U' = U_P$. That is, $U_P + K_P = (U + K)_P = W_P$ and $U_P \cap K_P = (U \cap K)_P \leq Rad(W_P)$ by **Corollary 2.1**. By **[1, Corollary 3.26]** and P**roposition 2.3**, we obtain $U + K = W$ and $U \cap K \leq Rad(W)$. Hence K has a weak $Rad-$ supplement in W. This implies that, M is a completely weak $Rad-$supplemented module.

Now, we give the following corollary to the **Proposition 3.3**.

Corollary 3.4. Let M be an $R-$module and P a maximal ideal of R such that for each proper submodule K of M we have $S(K) \subseteq P$. Let $M = N \oplus L$, where N, L are submodules of M. If M_P is a completely weak $Rad-$supplemented R_P-module, then N and L is also a completely weak $Rad-$supplemented $R-$module.

Proof. The proof follows directly by **Proposition 3.3** and **[9, Proposition 2.2].**

Now, we give a condition under which, we can extend the result of **[10, Lemma 3]**, to the localized modules.

Lemma 3.5. Let M be an $R-$module and P a maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. Let $M = U + V$ for submodules U and V of M. If V_P contains a $Rad-$supplement submodule of U_P in M_P, then $U \cap V$ has a $Rad-$supplement submodule in V.

Proof. Let K be a submodule of V, and suppose that a submodule K_P of V_P is a $Rad-$ supplement of U_P in M_P, then we have $U_P + K_P = M_P$ and $U_P \cap K_P \leq (Rad\, K_P)$, from the modular law, we have $U_P \cap V_P + K_P = V_P$, since $K_P \leq V_P$, then $(U_P \cap V_P) \cap K_P = U_P \cap K_P \leq (RadK_P)$, by **Corollary 2.1**, we get $((U \cap V) + K)_P = V_P$ and $((U \cap V) \cap K)_P \leq (RadK_P)$ and by **[1, Corollary 3.26]**, we have $Rad(K_P) = (RadK)_P$, hence by **Corollary 2.3** and **Proposition 2.2**, we get that $(U \cap V) + K = V$ and $(U \cap V) \cap K \leq RadK$. Thus K is a $Rad-$supplement submodule of $(U \cap V)$ in V.

Now, we give a condition under which we can extend the result of **[11, Lemma 4]**, to the localized modules.

Corollary 3.6. Let M be an $R-$module and V a $Rad-$supplement submodule of U in M, let P be any maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If $(U \cap V)_P$ is a supplement submodule in U_P, then V is supplement submodule of some submodule in M.

Proof. Let K be a submodule of U and $(U \cap V)_P$ is a supplement submodule of K_P in U_P. Then we have $K_P + (U \cap V)_P = U_P$ and $K_P \cap (U \cap V)_P \ll (U \cap V)_P$, then by **Corollary 2.1**, we get $(K + (U \cap V))_P = U_P$ and $(K \cap (U \cap V))_P \ll (U \cap V)_P$, hence by **Corollary 2.3** and **[1, Corollary 3.26]**, we get $K + (U \cap V) = U$ and $K \cap (U \cap V) \ll (U \cap V)$, it follows that $(U \cap V)$ is supplement submodule of K in U. Thus by **Lemma 2.4**, we get that V is supplement submodule of some submodule of M.

In **[11, Proposition 6]**, it is proved that, if every Rad −supplement submodule of a module is $Rad-$ supplemented module, then the module itself is a supplemented module and now we extend this fact to the localized module.

Proposition 3.7. Let M be a reduced module and P a maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If every Rad −supplement submodule of M_P is Rad −supplemented, then M is a supplemented module.

Proof. Let U and V be submodules of M and V_P a Rad −supplemented submodule of U_P in M_P. Then we have $U_P + V_P = M_P$ and $U_P \cap V_P \leq RadV_P$, since M is a reduced $R-$ module, then by **[1, Corollary 3.26]**, we get M_P is a reduced R_P −module, and we have V_P is a Rad −supplemented, then by **[11, Proposition 5]**, we get $RadV_P \ll V_P$, hence $U_P + V_P = M_P$ and $U_P \cap V_P \leq RadV_P \ll V_P$, then by **Corollary 2.1**, we get $(U + V)_P = M_P$ and $(U \cap V)_P \ll V_P$, hence by **Corollary 2.3** and **[1, Corollary 3.26]**, we get $U + V = M$ and $U \cap V \ll V$, it follows that V is a supplement submodule of U in M. Thus M is supplement R −module.

In the following result, we prove that if a module is a sum of two of its submodules for which the localization of one of them is supplemented, then the submodule contains a supplement of the other submodule.

Corollary 3.8. Let M be an R −module with submodules U and $V$of M, and let P be any maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$, suppose that $M = U + V$. If V_P is a supplemented R_P −module, then V contains a supplement submodule of U in M.

Proof. Let L be a submodule of V and let L_P be a supplement of $U_P \cap V_P$ in V_P. Then, we have $L_P + (U_P \cap V_P) = V_P$ and $L_P \cap (U_P \cap V_P) \ll L_P$, where $U_P \cap L_P = L_P \cap (U_P \cap V_P) \ll L_P$, hence by **Corollary 2.1**, we get $(L + (U + V))_P = V_P$ and $(U \cap L)_P = (L \cap (U \cap V))_P \ll V_P$ and by **Corollary 2.3** and **[1, Corollary 3.26]**, we get that $L + (U \cap V) = V$ and $U \cap L = L \cap (U \cap V) \ll L$, this means that L is a supplement of $(U \cap V)$ in V. Now, $M = U + V = U + (U \cap V) + L = U + L$, hence we get $M = U + L$ and $U \cap L \ll L$, it follows that L is a supplement of U in M. Thus V contains a supplement of U in M.

In **[10, proposition 5]**, it is proved that if a module is amply Rad −supplemented, then it is a hollow radical module. Now, we give a condition under which, we can extend this result to the localized module.

Proposition 3.9. Let R be a Noetherian ring and M be a simply radical R −module. Let P be a maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If M_P is an amply Rad −supplemented R_P −module, then M is hollow radical R −module.

Proof. Let U be a submodule of M and suppose that $U_P + V' = M_P$ for a submodule V' of M_P, by **[1, Lemma 3.16]**, there exists a submodule $V \leq M$ such that $V' = V_P$, hence we get $U_P + V_P = M_P$. By hypothesis there exists a submodule L' of V_P such that $U_P + L' = M_P$ and $U_P \cap L' \leq Rad\,(L')$, again by **[1, Lemma 3.16]**, there exists a submodule $L \leq V$ such that $L' = L_P$, hence $U_P + L_P = M_P$ and $U_P \cap L_P \leq Rad\,(L_P)$, by **Corollary 2.1**, $(U + L)_P = M_P$ and $(U \cap L)_P \leq Rad(L_P)$ and by **[1, Corollary 3.26]**, we have $Rad(L_P) = (RadL)_P$, hence by **Corollary 2.3** and **Proposition 2.2**, we get that $M = U + L$ and $U \cap L \leq Rad(L)$, and since M is simply radical, it follows that $Rad(L) = L \cap Rad(M) = L \cap M = L$, so L is a radical submodule, therefore $L = M$ and so $V = M$. Hence, we deduce that U is a small submodule in M. Hence, M is a hollow radical R −module.

Next, we prove that if every submodule of a localized module is Rad −supplemented, then the module itself is amply Rad −supplemented.

Proposition 3.10. Let M be an R −module and P a maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If every submodule of M_P is a Rad −supplemented R_P −module, then M is an amply Rad −supplemented R −module.

Proof. Let N be a submodule of M and $L' \leq M_P$**.** Then $N_P \leq M_P$ and by **[1, Lemma 3.16]**, there exists a submodule L of M such that $L' = L_P$, suppose that $M_P = N_P + L_P$, by assumption there exists a submodule H' of L_P such that $(N_P \cap L_P) + H' = L_P$ and $(N_P \cap L_P) \cap H' = N_P \cap H' \leq \mathrm{Rad}H'$, again by **[1, Lemma 3.16]**, there exists a submodule $H \leq L$ such that $H' = H_P$, hence $(N_P \cap L_P) + H_P = L_P$ and $(N_P \cap L_P) \cap H_P = N_P \cap H_P \leq \mathrm{Rad}(H_P)$. Thus, $L_P = H_P + (N_P \cap L_P) \leq N_P + H_P$ and hence $M_P = N_P + L_P \leq N_P + H_P$. Therefore, $M_P = N_P + H_P$ and $N_P \cap H_P \leq \mathrm{Rad}(H_P)$, by **[1, Corollary 3.26]**, we have $Rad(H_P) = (RadH)_P$, hence by **Corollary 2.1**, we get $M_P = (N + H)_P$ and $(N \cap H)_P \leq (\mathrm{Rad}H)_P$ and by **Corollary 2.3** and **Proposition 2.2**, we get $M = N + H$ and $N \cap H = RadH$. Hence N has a Rad −supplement $H \leq L$. Thus M is an amply Rad −supplemented module.

In **[13, Proposition 2.5]**, it is proved that if a module is a sum of two Rad −supplemented submodules, then the module itself is Rad −supplemented. Now, we extend this result to the localized module.

Corollary 3.11. Let N and L be $Rad-$supplemented $R-$modules and P a maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If $M_P = N_P + L_P$, then M is a $Rad-$supplemented $R-$module.

Proof. Since, $M_P = N_P + L_P$, then by **Corollary 2.1**, we have $M_P = (N + L)_P$ and also by **Corollary 2.3**, we get $M = N + L$. Hence by **[13, Proposition 2.5]**, we get that M is a $Rad-$supplemented module.

In **[13, Proposition 2.1]**, it is proved that a $Rad-$supplemented module which has zero Jacobson radical is semi simple. Now, we prove this result for the localized module.

Corollary 3.12. Let M be a $Rad-$supplemented $R-$module with a submodule L of M and P a prime ideal of R such that for each proper submodule K of M we have $S(K) \subseteq P$. If $L_P \cap RadM_P = 0$, then L is semi simple.

Proof. Since $L_P \cap RadM_P = 0$ and by **[1, Corollary 3.26]**, we have $Rad(M_P) = (RadM)_P$, then by **Corollary 2.1**, we get $(L \cap RadM)_P = 0$ and by **[7, Corollary 2.3]**, we get $L \cap RadM = 0$, hence by **[13, Proposition 2.1]**, we get that L is semi simple.

In **[13, Proposition 3.2]**, it is proved that every supplement submodule of a weak $Rad-$supplemented module is also weak $Rad-$supplemented. Now, we prove this result for the localized module.

Proposition 3.13. Let M be an $R-$module and P a maximal ideal of R such that for each proper submodule K of M, we have $S(K) \subseteq P$. If M_P is a weak $Rad-$supplemented $R-$module, then every supplement submodule of M is weak $Rad-$supplemented $R-$module.

Proof. Let K be a supplement submodule of M. For any submodule $N \leq K$, since M_P is a weak $Rad-$supplemented module, then there exists $L' \leq M_P$ such that $M_P = N_P + L'$ and $N_P \cap L' \leq RadM_P$, by **[1, Lemma 3.16]**, there exists a submodule $L \leq M$ such that $L' = L_P$, hence we get $M_P = N_P + L_P$ and $N_P \cap L_P \leq Rad(M_P)$. Thus, $K_P = K_P \cap M_P = K_P \cap (N_P + L_P) = N_P + (K_P \cap L_P)$ and $N_P \cap (K_P \cap L_P) = K_P \cap (N_P \cap L_P) \leq K_P \cap Rad(M_P) = Rad(K_P)$ by **[13, Lemma 1.1].** Hence, $N_P + (K_P \cap L_P) = K_P$ and $N_P \cap (K_P \cap L_P) \leq Rad(K_P)$, by **[1, Corollary 3.26]**, we have $Rad(K_P) = (RadK)_P$, hence by **Corollary 2.1**, we get $(N + (K \cap L))_P = K_P$ and $(N \cap (K \cap L))_P \leq (RadK)_P$, hence by **Corollary 2.3**, and **Proposition 2.2**, we get that $N + (K \cap L) = K$ and $N \cap (K \cap L) \leq RadK$. Therefore, we get that K is a weak $Rad-$supplemented $R-$module.

Next, we prove that, if the sum of the localization of two submodules of a module has a $Rad-$supplement submodule and if one of the submodules is $Rad-$ supplemented, then the other submodule has a $Rad-$ supplement submodule.

Proposition 3.14. Let M be an R −module with submodules $U,V \leq M$, let U be a Rad −supplemented module and P a maximal ideal of R such that for each proper submodule K of M we have $S(K) \subseteq P$. If $U_P + V_P$ has a Rad − supplement submodule in M_P, then V has a Rad −supplement submodule in M.

Proof. Let L be a submodule of V. Since, $U_P + V_P$ has a Rad −supplement in M_P , suppose that L_P is a Rad − supplement of $U_P + V_P$, hence we get $L_P + (U_P + V_P) = M_P$ and $L_P \cap (U_P + V_P) \leq Rad(L_P)$, by **[1, Corollary 3.26]**, we have $Rad(L_P) = (RadL)_P$, hence by **Corollary 2.1**, we get $(L + (U + V))_P = M_P$ and $(L \cap (U + V))_P \leq (RadL)_P$, and by **Corollary 2.3** and **Proposition 2.2**, we get that $L + (U + V) = M$ and $L \cap (U + V) \leq Rad(L)$. For $(L + V) \cap U$, since U is a Rad −supplemented module, there exists $K \leq U$ such that $(L + V) \cap U + K = U$ and $(L + V) \cap K \leq RadK$. Thus we have $L + V + K = M$ and $(L + V) \cap K \leq RadK$, that is K is a Rad −supplement of $L + V$ in M . It is clear that $(L + K) + V = M$, since $K + V \leq U + V$, $L \cap (K + V) \leq L \cap (U + V) \leq RadL$, we get that $(L + K) \cap V \leq L \cap (K + V) + K \cap (L + V) \leq RadL + RadK \leq Rad(L + K)$. Hence we get $(L + K) + V = M$ and $(L + K) \cap V \leq Rad(L + K)$, that means $L + K$ is a Rad − supplemented submodule of V in M . Thus V has a Rad −supplement submodule in M.

Reference

[1] Akray, I., Jabbar, A. K. and Sazeedeh, R.: Some Conditions Under Which Certain Types of Modules Possess Localization Property, Journal Duhok university, 15(1), 2012, 144-152.

[2] Atani, S. E. and Darani, A. Y. : Notes on the Primal Submodules, Chiang Mai J. Sci., 35(3), (2008), 399-410.

[3] Atiiyah, M. F. and Macdonald, I. G.: Introduction to Commutative Algebra, Addison-Wesley Publishing Company, 1969.

[4] Bilhan. G.: Amply FWS modules, Journal of Arts and Sciences, Sayi : 4, Aralik 2005.

[5] Clark, J., Lomp, C., Vajana, N. and Wisbauer R.:. Lifting Modules. Birkhauser Verlag Basel, Boston-berlin, 2006.

[6] Jabbar, A. K. and Hassan, N. H. : On Locally Multiplication Modules, International Mathematical Forum, 11(5), 2016, 213-226.

[7] Jabbar, A. K.: A Generalization of Prime and Weakly Prime submodules, Pure Mathematical sciences, 2(1), 2013, 1-11.

[8] Nebiev, C.: Amply weak supplemented modules, International Journal Computational Cogintion, 3(1), 2005, 88-90.

[9] NISANCI, B., TURKMEN, E. and PANCAR, A.: Completely Weak Rad-supplemented Modules, International Journal of Computational Cognttion, 7(2), 2009.

[10] Turkmen, E. and Pancar, A.: Characterizations of Rad-supplemented modules, Miskolc Mathematical Notes, 13(2), 2012, 569 – 580.

[11] Turkmen, E. and Pancar, A.: Some properties of Rad-supplemented modules, International Journal of the Physical Sciences, 6(35), 2011, 7904 – 7909.

[12] Turkmen, E. and Pancer, A.: On Radical Supplemented Modules, International Journal of Computational Cognttion, 7(1), 2009.

[13] Wang, Y. and Ding, N.: Generalized Supplemented Modules, Taiwanese Journal of Mathematics, 10(6), 1589-1601, 2006.