On $\text{Rad} - \text{Supplemented Modules}$, $\text{Weak Rad} - \text{Supplemented Modules}$ and $\text{Completely Weak Rad} - \text{Supplemented Modules}$

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Abstract

In this paper some results that concerning localization of modules are proved. It also, studies the effect of localization on certain types of modules such as $\text{Rad} - \text{supplemented modules}$, $\text{weak Rad} - \text{supplemented modules}$ and $\text{completely weak Rad} - \text{supplemented modules}$. Several conditions are given under which certain properties of such types of algebraic structures are preserved under localization.

Keywords: Localization of modules, $\text{Rad} - \text{supplemented modules}$, $\text{weak Rad} - \text{supplemented modules}$, $\text{completely weak Rad} - \text{supplemented modules}$ and $\text{amply supplemented modules}$.

1 Introduction

Throughout this paper, $R$ is a commutative ring with identity and $M$ is a unitary left $R$-module. By $N \leq M$ we mean $N$ is a submodule of $M$. A submodule $V$ of $M$ is called a small submodule of $M$, denoted by $V \ll M$, if $L \leq M$ is any submodule such that $V + L = M$, then $L = M$ [12], and $V$ is called a supplement (a weak supplement) of $U \leq M$ if $M = U + V$ and $U \cap V \ll V$ ($U \cap V \ll M$) [8]. Moreover, $M$ is called a supplemented (a weak supplemented) module if every submodule of $M$ has a supplement (a weak supplement) in $M$ [4]. $M$ is called an amply supplemented $R$-module if for any submodules $U$ and $V$ of $M$ with $M = U + V$ there exists a submodule $K$ of $U$ such that $K \leq V$ [4], and $M$ is called an amply weak supplemented $R$-module if every submodule of $M$ has amply supplement in $M$ [8] and $V$ is called a $\text{Rad}$-supplement or a generalized supplement (a weak $\text{Rad}$-supplement or a generalized weak supplement) of $U$ in $M$ if $M = U + V$ and $U \cap V \leq \text{Rad}V$ ($U \cap V \leq \text{Rad}M$) [9]. Moreover, $M$ is called a $\text{Rad}$-supplemented or a generalized supplemented (a weakly $\text{Rad}$-supplemented or a generalized weakly supplemented) module if every submodule of $M$ has a $\text{Rad}$-supplement or a generalized supplement (a weak $\text{Rad}$-supplement or a generalized weak supplement) in $M$ [11]. $M$ is called completely weak $\text{Rad}$-supplemented if every submodule of $M$ is weakly $\text{Rad}$-supplemented [9] and it is called amply $\text{Rad}$-supplemented (or generalized amply supplemented) in case $M = U + V$ implies that $U$ has a $\text{Rad}$-supplement.
(or has a generalized supplement) in $V$ [13]. If $r \in R$, then we define $N; r = \{x \in M \mid rx \in N\}$ [1]. A submodule $K \leq M$ is called a radical submodule if $Rad(K) = K$. $M$ is called a hollow module if every proper submodule of $M$ is small in $M$ [13], and by a hollow radical submodule is meant a submodule which is both hollow and radical. $M$ is called a semi-simple if every submodule of $M$ is a direct summand [5]. For a submodule $K$ of $M$ we define $S_{M}(K) = \{r \in R \mid rx \in K\}$ for some $x \notin K$ [2]. A non-empty subset $S$ of $R$ is called a multiplicative system in $R$, if $0 \notin S$ and $a, b \in S$ implies $ab \in S$ [6]. If $S$ is a multiplicative system in $R$, then one can obtain an $R_{S}$ − module, denoted by $M_{S}$, under the module operations $x + y = \frac{tx + sy}{st}$ and $\frac{r}{s}x = \frac{rx}{us}$, for $r \in R_{S}$ and $\frac{x}{s}, \frac{y}{t} \in M_{S}$, so that when we say $M_{S}$ is a module we mean $M_{S}$ is an $R_{S}$ − module. In fact, this module $M_{S}$ is known as the localization of $M$ at the multiplicative system $S$ [1].

2. Some Basic Preliminaries

The following are some known results on which we depend to prove the main results of this paper.

**Corollary 2.1.** [3] Let $M$ be an $R$ − module and $P$ a prime ideal of $R$. For submodules $N, L$ of $M$ the following are satisfied.

1. $(M/N)_{P} \cong M_{P}/N_{P}$.
2. $(N + L)_{P} \cong N_{P} + L_{P}$.
3. $(N \cap L)_{P} \cong N_{P} \cap L_{P}$.

**Proposition 2.2.** [7] Let $L$ and $N$ be submodules of an $R$ − module $M$. Then $L \subseteq N$ if and only if $L_{P} \subseteq N_{P}$ for every maximal ideal $P$ of $R$.

**Corollary 2.3.** [7] Let $L$ and $N$ be submodules of an $R$ − module $M$. Then $L = N$ if and only if $L_{P} = N_{P}$ for every maximal ideal $P$ of $R$. In particular, $N = M$ if and only if $N_{P} = M_{P}$, for every maximal ideal $P$ of $R$.

**Lemma 2.4.** [11] Let $M$ be an $R$ − module and $V$ a $Rad$ − supplement submodule of $U$ in $M$. If $U \cap V$ is a supplement submodule in $U$, then $V$ is supplement submodule in $M$.

3. Main Results

First, we give an example of an $R$ − module $M$ in which there is a prime ideal $P$ of $R$ and for which $S(K) \supseteq P$, for all proper submodules $K$ of $M$.

**Example.** Consider $Z_{8}$ as a $Z$ − module, that is take $R = Z$ and $M = Z_{8}$. Clearly $P = \langle 2 \rangle$ is a prime ideal of $Z$. The only proper submodules of $Z_{8}$ are $\{0\}, \{0, 4\}, \{0, 2, 4, 6\}$. Now, one can easily calculate $S(\{0\}), S(\{0, 4\})$ and $S(\{0, 2, 4, 6\})$ and get that $S(\{0\}) = \langle 2 \rangle = P \subseteq P$. 

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\(S(\{0,4\}) = \langle 4 \rangle \subseteq P.\)
\(S(\{0,2,4,6\}) = \langle 8 \rangle \subseteq P.\)
Hence, \(Z_8\) is a \(Z\) – module, where \(P = \langle 2 \rangle\) is a prime ideal of \(Z\) and such that \(S(K) \subseteq P\) for all submodules \(K\) of \(Z_8\).

Now, we prove the first result.

Lemma 3.1. Let \(M\) be an \(R\) – module with submodules \(U\) and \(V\) of \(M\) and \(P\) be any maximal ideal of \(R\) such that for each proper submodule \(K\) of \(M\), we have \(S(K) \subseteq P\). If \(V_P\) is a weak \(Rad\) – supplement submodule of \(U_P\) in \(M_P\), then for a submodule \(L'\) of \(U_P\), there exists a submodule \(L\) of \(U\) such that \(\frac{V + L}{L}\) is a weak \(Rad\) – supplement submodule of \(\frac{U}{L}\) in \(\frac{M}{L}\).

Proof. As \(V_P\) is a weak \(Rad\) – supplement submodule of \(U_P\) in \(M_P\), we have \(U_P + V_P = M_P\) and \(U_P \cap V_P \subseteq Rad(M_P)\), this implies that \((U + V)_P = M_P\) and \((U \cap V)_P \subseteq Rad(M_P)\). By using [1, Corollary 3.26], we have \((U \cap V)_P \leq (RadM)_P\), so by Corollary 2.3 and Proposition 2.2, we get \(U + V = M\) and \(U \cap V \leq RadM\), so that \(V\) is a weak \(Rad\) – supplement submodule of \(U\) in \(M\). Now, since \(L' \leq U_P\), so by [1, Lemma 3.16], there exists a submodule \(L \leq U\) such that \(L' = L_P\). Thus by [9, Lemma 2.1], we get \(\frac{V + L}{L}\) is a weak \(Rad\) – supplement submodule of \(\frac{U}{L}\) in \(\frac{M}{L}\).

In \([9, Proposition II.1]\), it is proved that every factor module of completely weak \(Rad\) – supplemented module is also a completely weak \(Rad\) – supplemented module. Now, we prove this result by replacing the module with its localization at maximal ideals.

Proposition 3.2. Let \(M\) be an \(R\) – module with a submodule \(L\) of \(M\) and \(P\) be any maximal ideal of \(R\) such that for each proper submodule \(K\) of \(M\) we have \(S(K) \subseteq P\). If \(M_P\) is a completely weak \(Rad\) – supplemented \(R_P\) – module, then \(\frac{M}{L}\) is a completely weak \(Rad\) – supplemented \(R\) – module.

Proof. Let \(\frac{K}{L}\) be a submodule of \(\frac{M}{L}\), where \(L \leq K \leq M\). Then, \(\frac{K}{L}\) is a submodule of \(\frac{M}{L}\), where \(L_P \leq K_P \leq M_P\). Let \(\frac{U_P}{L_P}\) be a submodule of \(\frac{K}{L}\), where \(L_P \leq U_P \leq K_P\).

Since \(M_P\) is a completely weak \(Rad\) – supplemented \(R_P\) – module, there exists a submodule \(V_P\) of \(K_P\) for which \(U_P + V_P = K_P\) and \(U_P \cap V_P \leq Rad(K_P)\). As \(V_P\) is a weak \(Rad\) – supplement of \(U_P\) in \(K_P\) and \(L_P \leq U_P\), so by Lemma 3.1, we get that \(\frac{V + L}{L}\) is a weak \(Rad\) – supplement of \(\frac{U}{L}\) in \(\frac{K}{L}\). Hence \(\frac{M}{L}\) is a weakly \(Rad\) – supplemented module. So that, \(\frac{M}{L}\) is a completely weak \(Rad\) – supplemented \(R\) – module.

In the next result, we prove that, if the localization of a module at a maximal ideal is completely weak \(Rad\) – supplemented, then the module itself is so.
Proposition 3.3. Let $M$ be an $R$ module and $P$ be any prime ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$. If $M_P$ is a completely weak $Rad$ supplemented $R_P$ module, then $M$ is also a completely weak $Rad$ supplemented $R$ module.

Proof. Let $W$ be any submodule of $R$ module $M$ and $K$ be any submodule of $W$. Then, by Proposition 2.2, $W_P$ is a submodule of $M_P$ and $K_P$ is a submodule of $W_P$. Since $M_P$ is completely weak $Rad$ supplemented, then, $W_P$ is weak $Rad$ supplemented, therefore, $K_P$ has a weak $Rad$ supplemented in $M_P$. This implies that, there exists a submodule $U'$ of $W_P$ such that $U' + K_P = W_P$ and $U' \cap K_P \leq Rad(W_P)$. By [1, Lemma 3.16], there exists a submodule $U$ of $W$ such that $U' = U_P$. That is, $U_P + K_P = (U + K)_P = W_P$ and $U_P \cap K_P = (U \cap K)_P \leq Rad(W_P)$ by Corollary 2.1. By [1, Corollary 3.26] and Proposition 2.3, we obtain $U + K = W$ and $U \cap K \leq Rad(W)$. Hence $K$ has a weak $Rad$ supplement in $W$. This implies that, $M$ is a completely weak $Rad$ supplemented module.

Now, we give the following corollary to the Proposition 3.3.

Corollary 3.4. Let $M$ be an $R$ module and $P$ a maximal ideal of $R$ such that for each proper submodule $K$ of $M$ we have $S(K) \subseteq P$. Let $M = N \oplus L$, where $N, L$ are submodules of $M$. If $M_P$ is a completely weak $Rad$ supplemented $R_P$ module, then $N$ and $L$ is also a completely weak $Rad$ supplemented $R$ module.

Proof. The proof follows directly by Proposition 3.3 and [9, Proposition 2.2].

Now, we give a condition under which, we can extend the result of [10, Lemma 3], to the localized modules.

Lemma 3.5. Let $M$ be an $R$ module and $P$ a maximal ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$. Let $M = U + V$ for submodules $U$ and $V$ of $M$. If $V_P$ contains a $Rad$ supplement submodule of $U_P$ in $M_P$, then $U \cap V$ has a $Rad$ supplement submodule in $V$.

Proof. Let $K$ be a submodule of $V$, and suppose that a submodule $K_P$ of $V_P$ is a $Rad$ supplement of $U_P$ in $M_P$, then we have $U_P + K_P = M_P$ and $U_P \cap K_P \leq (Rad K_P)$, from the modular law, we have $U_P \cap V_P + K_P = V_P$, since $K_P \leq V_P$, then $(U_P \cap V_P) \cap K_P = U_P \cap K_P \leq (Rad K_P)$, by Corollary 2.1, we get $((U \cap V) + K)_P = V_P$ and $((U \cap V) \cap K)_P \leq (Rad K_P)$ and by [1, Corollary 3.26], we have $Rad(K_P) = (Rad K)_P$, hence by Corollary 2.3 and Proposition 2.2, we get that $(U \cap V) + K = V$ and $(U \cap V) \cap K \leq Rad K$. Thus $K$ is a $Rad$ supplement submodule of $(U \cap V)$ in $V$.

Now, we give a condition under which we can extend the result of [11, Lemma 4], to the localized modules.

Corollary 3.6. Let $M$ be an $R$ module and $V$ a $Rad$ supplement submodule of $U$ in $M$, let $P$ be any maximal ideal of $R$ such that for each proper submodule $K$ of $M$,
we have $S(K) \subseteq P$. If $(U \cap V)_P$ is a supplement submodule in $U_P$, then $V$ is supplement submodule of some submodule in $M$.

**Proof.** Let $K$ be a submodule of $U$ and $(U \cap V)_P$ is a supplement submodule of $K_P$ in $U_P$. Then we have $K_P + (U \cap V)_P = U_P$ and $K_P \cap (U \cap V)_P \ll (U \cap V)_P$, then by Corollary 2.1, we get $(K + (U \cap V))_P = U_P$ and $(K \cap (U \cap V))_P \ll (U \cap V)_P$, hence by Corollary 2.3 and [1, Corollary 3.26], we get $K + (U \cap V) = U$ and $K \cap (U \cap V) \ll (U \cap V)$, it follows that $(U \cap V)$ is supplement submodule of $K$ in $U$. Thus by Lemma 2.4, we get that $V$ is supplement submodule of some submodule of $M$.

In [11, Proposition 6], it is proved that, if every $Rad$ —supplement submodule of a module is $Rad$ —supplemented module, then the module itself is a supplemented module and now we extend this fact to the localized module.

**Proposition 3.7.** Let $M$ be a reduced module and $P$ a maximal ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$. If every $Rad$ —supplement submodule of $M_P$ is $Rad$ —supplemented, then $M$ is a supplemented module.

**Proof.** Let $U$ and $V$ be submodules of $M$ and $V_P$ a $Rad$ —supplemented submodule of $U_P$ in $M_P$. Then we have $U_P + V_P = M_P$ and $U_P \cap V_P \leq RadV_P$, since $M$ is a reduced $R$ —module, then by [1, Corollary 3.26], we get $M_P$ is a reduced $R_P$ —module, and we have $V_P$ is a $Rad$ —supplemented, then by [11, Proposition 5], we get $RadV_P \ll V_P$, hence $U_P + V_P = M_P$ and $U_P \cap V_P \leq RadV_P \ll V_P$, then by Corollary 2.1, we get $(U + V)_P = M_P$ and $(U \cap V)_P \ll V_P$, hence by Corollary 2.3 and [1, Corollary 3.26], we get $U + V = M$ and $U \cap V \ll V$, it follows that $V$ is a supplement submodule of $U$ in $M$. Thus $M$ is supplement $R$ —module.

In the following result, we prove that if a module is a sum of two of its submodules for which the localization of one of them is supplemented, then the submodule contains a supplement of the other submodule.

**Corollary 3.8.** Let $M$ be an $R$ —module with submodules $U$ and $V$ of $M$, and let $P$ be any maximal ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$, suppose that $M = U + V$. If $V_P$ is a supplemented $R_P$ —module, then $V$ contains a supplement submodule of $U$ in $M$.

**Proof.** Let $L$ be a submodule of $V$ and let $L_P$ be a supplement of $U_P \cap V_P$ in $V_P$. Then, we have $L_P + (U_P \cap V_P) = V_P$ and $L_P \cap (U_P \cap V_P) \ll L_P$, where $U_P \cap L_P = L_P \cap (U_P \cap V_P) \ll L_P$, hence by Corollary 2.1, we get $(L + (U + V))_P = V_P$ and $(U \cap L)_P = (L \cap (U \cap V))_P \ll V_P$, and by Corollary 2.3 and [1, Corollary 3.26], we get that $L + (U \cap V) = V$ and $U \cap L = L \cap (U \cap V) \ll L$, this means that $L$ is a supplement of $(U \cap V)$ in $V$. Now, $M = U + V = U + (U \cap V) + L = U + L$, hence we get $M = U + L$ and $U \cap L \ll L$, it follows that $L$ is a supplement of $U$ in $M$. Thus $V$ contains a supplement of $U$ in $M$. 
In [10, proposition 5], it is proved that if a module is amply $\text{Rad}$-supplemented, then it is a hollow radical module. Now, we give a condition under which, we can extend this result to the localized module.

**Proposition 3.9.** Let $R$ be a Noetherian ring and $M$ be a simply radical $R$-module. Let $P$ be a maximal ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$. If $M_P$ is an amply $\text{Rad}$-supplemented $R_P$-module, then $M$ is hollow radical $R$-module.

**Proof.** Let $U$ be a submodule of $M$ and suppose that $U_P + V' = M_P$, for a submodule $V'$ of $M_P$, by [1, Lemma 3.16], there exists a submodule $V \leq M$ such that $V' = V_P$, hence we get $U_P + V_P = M_P$. By hypothesis there exists a submodule $L'$ of $V_P$ such that $U_P + L' = M_P$ and $U_P \cap L_P \leq \text{Rad}(L)$, again by [1, Lemma 3.16], there exists a submodule $L \leq V$ such that $L' = L_P$, hence $U_P + L_P = M_P$ and $U_P \cap L_P \leq \text{Rad}(L_P)$, by Corollary 2.1, $(U + L)_P = M_P$ and $(U \cap L)_P \leq \text{Rad}(L_P)$ and by [1, Corollary 3.26], we have $\text{Rad}(L_P) = (\text{Rad}L)_P$, hence by Corollary 2.3 and Proposition 2.2, we get that $M = U + L$ and $U \cap L \leq \text{Rad}(L)$, and since $M$ is simply radical, it follows that $\text{Rad}(L) = L \cap \text{Rad}(M) = L \cap M = L$, so $L$ is a radical submodule, therefore $L = M$ and so $V = M$. Hence, we deduce that $U$ is a small submodule in $M$. Hence, $M$ is a hollow radical $R$-module.

Next, we prove that if every submodule of a localized module is $\text{Rad}$-supplemented, then the module itself is amply $\text{Rad}$-supplemented.

**Proposition 3.10.** Let $M$ be an $R$-module and $P$ a maximal ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$. If every submodule of $M_P$ is a $\text{Rad}$-supplemented $R_P$-module, then $M$ is an amply $\text{Rad}$-supplemented $R$-module.

**Proof.** Let $N$ be a submodule of $M$ and $L' \leq M_P$. Then $N_P \leq M_P$ and by [1, Lemma 3.16], there exists a submodule $L$ of $M$ such that $L' = L_P$, suppose that $M_P = N_P + L_P$, by assumption there exists a submodule $H'$ of $L_P$ such that $(N_P \cap L_P) + H' = L_P$ and $(N_P \cap L_P) \cap H' = N_P \cap H' = \text{Rad}(H')$, again by [1, Lemma 3.16], there exists a submodule $H \leq L$ such that $H' = H_P$, hence $(N_P \cap L_P) + H = L_P$ and $(N_P \cap L_P) \cap H = N_P \cap H = \text{Rad}(H_P)$. Thus, $L_P = H_P + (N_P \cap L_P) \leq N_P + H_P$ and hence $M_P = N_P + L_P \leq N_P + H_P$. Therefore, $M_P = N_P + H_P$ and $N_P \cap H_P \leq \text{Rad}(H_P)$, by [1, Corollary 3.26], we have $\text{Rad}(H_P) = (\text{Rad}H)_P$, hence by Corollary 2.1, we get $M_P = (N + H)_P$ and $(N \cap H)_P \leq (\text{Rad}H)_P$ and by Corollary 2.3 and Proposition 2.2, we get $M = N + H$ and $N \cap H = \text{Rad}H$. Hence $N$ has a $\text{Rad}$-supplement $H \leq L$. Thus $M$ is an amply $\text{Rad}$-supplemented module.

In [13, Proposition 2.5], it is proved that if a module is a sum of two $\text{Rad}$-supplemented submodules, then the module itself is $\text{Rad}$-supplemented. Now, we extend this result to the localized module.
Corollary 3.11. Let $N$ and $L$ be $Rad$ -- supplemented $R$ -- modules and $P$ a maximal ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$. If $M_P = N_P + L_P$, then $M$ is a $Rad$ -- supplemented $R$ -- module.

**Proof.** Since $M_P = N_P + L_P$, then by **Corollary 2.1**, we have $M_P = (N + L)_P$ and also by **Corollary 2.3**, we get $M = N + L$. Hence by [13, Proposition 2.5], we get that $M$ is a $Rad$ -- supplemented module.

In [13, Proposition 2.1], it is proved that a $Rad$ -- supplemented module which has zero Jacobson radical is semi simple. Now, we prove this result for the localized module.

Corollary 3.12. Let $M$ be a $Rad$ -- supplemented $R$ -- module with a submodule $L$ of $M$ and $P$ a prime ideal of $R$ such that for each proper submodule $K$ of $M$ we have $S(K) \subseteq P$. If $L_P \cap RadM_P = 0$, then $L$ is semi simple.

**Proof.** Since $L_P \cap RadM_P = 0$ and by [1, Corollary 3.26], we have $Rad(M_P) = (RadM)_P$, then by **Corollary 2.1**, we get $(L \cap RadM)_P = 0$ and by [7, Corollary 2.3], we get $L \cap RadM = 0$, hence by [13, Proposition 2.1], we get that $L$ is semi simple.

In [13, Proposition 3.2], it is proved that every supplement submodule of a weak $Rad$ -- supplemented module is also weak $Rad$ -- supplemented. Now, we prove this result for the localized module.

**Proposition 3.13.** Let $M$ be an $R$ -- module and $P$ a maximal ideal of $R$ such that for each proper submodule $K$ of $M$, we have $S(K) \subseteq P$. If $M_P$ is a weak $Rad$ -- supplemented $R$ -- module, then every supplement submodule of $M$ is weak $Rad$ -- supplemented $R$ -- module.

**Proof.** Let $K$ be a supplement submodule of $M$. For any submodule $N \leq K$, since $M_P$ is a weak $Rad$ -- supplemented module, then there exists $L' \leq M_P$ such that $M_P = N_P + L'$ and $N_P \cap L'$ $\leq RadM_P$, by [1, Lemma 3.16], there exists a submodule $L \leq M$ such that $L' = L_P$, hence we get $M_P = N_P + L_P$ and $N_P \cap L_P \leq Rad(M_P)$. Thus, $K_P = K_P \cap M_P = K_P \cap (N_P + L_P) = N_P + (K_P \cap L_P)$ and $N_P \cap (K_P \cap L_P) = K_P \cap (N_P \cap L_P) \leq K_P \cap Rad(M_P) = Rad(K_P)$ by [13, Lemma 1.1]. Hence, $N_P + (K_P \cap L_P) = K_P$ and $N_P \cap (K_P \cap L_P) \leq Rad(K_P)$, by [1, Corollary 3.26], we have $Rad(K_P) = (RadK)_P$, hence by **Corollary 2.1**, we get $(N + (K \cap L))_P = K_P$ and $(N \cap (K \cap L))_P \leq (RadK)_P$, hence by **Corollary 2.3**, and **Proposition 2.2**, we get that $N + (K \cap L) = K$ and $N \cap (K \cap L) \leq RadK$. Therefore, we get that $K$ is a weak $Rad$ -- supplemented $R$ -- module.

Next, we prove that, if the sum of the localization of two submodules of a module has a $Rad$ -- supplement submodule and if one of the submodules is $Rad$ -- supplemented, then the other submodule has a $Rad$ -- supplement submodule.
Proposition 3.14. Let $M$ be an $R$-module with submodules $U, V \leq M$, let $U$ be a $Rad$-supplemented module and $P$ a maximal ideal of $R$ such that for each proper submodule $K$ of $M$ we have $S(K) \subseteq P$. If $U_P + V_P$ has a $Rad$-supplement submodule in $M_P$, then $V$ has a $Rad$-supplement submodule in $M$.

Proof. Let $L$ be a submodule of $V$. Since, $U_P + V_P$ has a $Rad$-supplement in $M_P$, suppose that $L_P$ is a $Rad$-supplement of $U_P + V_P$, hence we get $L_P + (U_P + V_P) = M_P$ and $L_P \cap (U_P + V_P) \leq Rad(L_P)$, by [1, Corollary 3.26], we have $Rad(L_P) = (RadL)_P$, hence by Corollary 2.1, we get $(L + (U + V))_P = M_P$ and $(L \cap (U + V))_P \leq (RadL)_P$, and by Corollary 2.3 and Proposition 2.2, we get that $L + (U + V) = M$ and $L \cap (U + V) \leq Rad(L)$. For $(L + V) \cap U$, since $U$ is a $Rad$-supplemented module, there exists $K \leq U$ such that $(L + V) \cap U + K = U$ and $(L + V) \cap K \leq RadK$. Thus we have $L + V + K = M$ and $(L + V) \cap K \leq RadK$, that is $K$ is a $Rad$-supplement of $L + V$ in $M$. It is clear that $(L + K) + V = M$, since $K + V \leq U + V$, $L \cap (K + V) \leq L \cap (U + V) \leq RadL$, we get that $(L + K) \cap V \leq L \cap (K + V) + K \cap (L + V) \leq RadL + RadK \leq Rad(L + K)$. Hence we get $(L + K) + V = M$ and $(L + K) \cap V \leq Rad(L + K)$, that means $L + K$ is a $Rad$-supplemented submodule of $V$ in $M$. Thus $V$ has a $Rad$-supplement submodule in $M$. 
Reference


