\( \lambda_{\beta c} \)-Connected Spaces and \( \lambda_{\beta c} \)-Components

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Abstract

In this paper, we define and study a new type of connected spaces called \( \lambda_{\beta c} \)-connected space. It is remarkable that the class of \( \lambda \)-connected spaces is a subclass of the class of \( \lambda_{\beta c} \)-connected spaces. We discuss some characterizations and properties of \( \lambda_{\beta c} \)-connected spaces, \( \lambda_{\beta c} \)-components and \( \lambda_{\beta c} \)-locally connected spaces.

1. Introduction

The study of semi-open sets and their properties was initiated by N. Levine [1] in 1963. In [2], S.F.Namiq defined an operation \( \lambda \) on the family of semi open sets in a topological space called s-operation via this operation, he defined \( \lambda \)-open sets. By using \( \lambda \)-open and semi closed set also S.F.Namiq in [3], defined \( \lambda_{\beta c} \)-open set and also investigated several properties of \( \lambda_{\beta c} \)-derived, \( \lambda_{\beta c} \)-interior and \( \lambda_{\beta c} \)-closure points in topological spaces, moreover In [4], S.F.Namiq defined \( \lambda \)-connected spaces by using \( \lambda \)-open sets. In [5], Furthermore S.F. Namiq defined \( \lambda_{c} \)-connected spaces via \( \lambda_{c} \)-open sets. S. Willard in [6], obtained some analogous properties of connectedness for \( \lambda_{\beta c} \)-connectedness. Throughout the present paper, a topological space is denoted by \((X, \tau)\) or simply by \(X\).
2. Preliminaries

First, we recall some definitions and results used in this paper. For any subset \( A \) of \( X \), the closure and the interior of \( A \) are denoted by \( Cl(A) \) and \( Int(A) \), respectively. A subset \( A \) of a space \( X \) is said to be semi open [1] if \( A \subseteq Cl(Int(A)) \). The complement of a semi open set is said to be semi closed [1]. The family of all semi open (resp. semi closed) sets in a space \( X \) is denoted by \( SO(X,\tau) \) or \( SO(X) \) (resp. \( SC(X,\tau) \) or \( SC(X) \)). A space \( X \) is said to be s-connected [7], if it is not the union of two nonempty disjoint semi open subsets of \( X \). A subset \( A \) of a topological space \( X \) is said to be \( \beta \)-open [8], if \( A \subseteq Cl(\lambda(Cl(A))) \). The complement of a \( \beta \)-open set is said to be \( \beta \)-closed. The family of all \( \beta \)-open (resp. \( \beta \)-closed) sets in a topological space \( (X,\tau) \) is denoted by \( \beta O(X,\tau) \) or \( \beta O(X) \) (resp. \( \beta C(X,\tau) \) or \( \beta C(X) \)). We consider \( \lambda:SO(X) \rightarrow P(X) \) as a function defined on \( SO(X) \) into the power set of \( X, P(X) \) and \( \lambda \) is called an \( s \)-operation if \( V \subseteq \lambda(V) \), for each semi open set \( V \). It is assumed that \( \lambda(\phi)=\phi \) and \( \lambda(X)=X \), for any \( s \)-operation \( \lambda \). Let \( X \) be a space and \( \lambda:SO(X) \rightarrow P(X) \) be an \( s \)-operation, then a subset \( A \) of \( X \) is called a \( \lambda \)-open set [2], which is equivalent to \( \lambda_{s} \)-open set [9], if for each \( x\in A \), there exists a semi open set \( U \) such that \( x\in U \) and \( \lambda(U) \subseteq A \). The complement of a \( \lambda \)-open set is said to be \( \lambda \)-closed. The family of all \( \lambda \)-open (resp., \( \lambda \)-closed) subsets of a space \( X \) is denoted by \( SO_{\lambda}(X,\tau) \) or \( SO_{\lambda}(X) \) (resp., \( SC_{\lambda}(X,\tau) \) or \( SC_{\lambda}(X) \)), then a \( \lambda \)-open subset \( A \) of \( X \) is called a \( \lambda_{c} \)-open set [2], if for each \( x\in A \), there exists a closed set \( F \) such that \( x\in F \subseteq A \). The family of all \( \lambda_{c} \)-open (resp., \( \lambda_{c} \)-closed) subsets of a space \( X \) is denoted by \( SO_{\lambda_{c}}(X,\tau) \) or \( SO_{\lambda_{c}}(X) \) (resp., \( SC_{\lambda_{c}}(X,\tau) \) or \( SC_{\lambda_{c}}(X) \)).
Now, we recall some definitions and restate some known results which will be used in the sequel.

**Definition 2.1[2].** Let $X$ be a space and $\lambda: SO(X) \to P(X)$ be an $s$-operation, then a subset $A$ of $X$ is called a $\lambda$-open set if for each $x \in A$ there exists a semi open set $U$ such that $x \in U$ and $\lambda(U) \subseteq A$.

The complement of a $\lambda$-open set is called $\lambda$-closed. The family of all $\lambda$-open (resp., $\lambda$-closed) subsets of a topological space $(X, \tau)$ is denoted by $SO_{\lambda}(X, \tau)$ or $SO_{\lambda}(X)$ (resp., $SC_{\lambda}(X, \tau)$ or $SC_{\lambda}(X)$).

**Definition 2.2[2].** A $\lambda$-open subset $A$ of $X$ is called a $\lambda_c$-open set if for each $x \in A$ there exists a closed set $F$ such that $x \in F \subseteq A$. The family of all $\lambda_c$-open (resp., $\lambda_c$-closed) subsets of a space $X$ is denoted by $SO_{\lambda_c}(X, \tau)$ or $SO_{\lambda_c}(X)$ (resp., $SC_{\lambda_c}(X, \tau)$ or $SC_{\lambda_c}(X)$).

**Definition 2.3[3].** A $\lambda$-open subset $A$ of $X$ is called a $\lambda_{\beta c}$-open set if for each $x \in A$, there exists a $\beta$-closed set $F$ such that $x \in F \subseteq A$. The family of all $\lambda_{\beta c}$-open (resp., $\lambda_{\beta c}$-closed) subsets of a space $X$ is denoted by $SO_{\lambda_{\beta c}}(X, \tau)$ or $SO_{\lambda_{\beta c}}(X)$ (resp., $SC_{\lambda_{\beta c}}(X, \tau)$ or $SC_{\lambda_{\beta c}}(X)$).

**Proposition 2.4[3].** For a space $X$, $SO_{\lambda_c}(X) \subseteq SO_{\lambda_{\beta c}}(X) \subseteq SO_{\beta}(X) \subseteq SO(X)$.

The following examples show that the converse of the above proposition may not be true in general.

**Example 2.5.** Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, X\}$. Define an $s$-operation $\lambda: SO(X) \to P(X)$ as follows:

$$\lambda(A) = \begin{cases} A & \text{if } b \in A \\ X & \text{otherwise} \end{cases}.$$

Here, we have $\{a, c\}$ is a semi open set, but it is not $\lambda$-open. And also we have $\{a, b\}$ is a $\lambda$-open set but it is not $\lambda_{\beta c}$-open set, but not $\lambda_c$-open.
Example 2.6. Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. We define an s-operation $\lambda : SO(X) \rightarrow P(X)$ as:

$$
\lambda(A) = \begin{cases} 
A & \text{if } A = \{b\} \\
X & \text{otherwise}
\end{cases}
$$

Here, we have $\{b\}$ is a $\lambda_{\beta c}$-open set, but it is not $\lambda_c$-open.

**Definition 2.17** [9]. Let $X$ be a space, an s-operation $\lambda$ is said to be $s$-regular if for every semi open sets $U$ and $V$ containing $x \in X$, there exists a semi open set $W$ containing $x$ such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.

**Definition 2.8** [3]. Let $X$ be a space and $A$ a subset of $X$. Then:

(1) The $\lambda_{\beta c}$-closure of $A$ ($\lambda_{\beta c} Cl(A)$) is the intersection of all $\lambda_{\beta c}$-closed sets which containing $A$.

(2) The $\lambda_{\beta c}$-interior of $A$ ($\lambda_{\beta c} Int(A)$) is the union of all $\lambda_{\beta c}$-open sets of $X$ which contained in $A$.

(3) A point $x \in X$ is said to be a $\lambda_{\beta c}$-limit point of $A$ if every $\lambda_{\beta c}$-open set containing $x$ contains a point of $A$ different from $x$, and the set of all $\lambda_{\beta c}$-limit points of $A$ is called the $\lambda_{\beta c}$-derived set of $A$, denoted by $\lambda_{\beta c} D(A)$.

**Proposition 2.9** [3]. For each point $x \in X$, $x \in \lambda_{\beta c} Cl(A)$ if and only if $V \cap A \neq \phi$, for every $V \in SO_{\lambda_{\beta c}}(X)$ such that $x \in V$.

**Proposition 2.10** [3]. Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of $\lambda_{\beta c}$-open sets in a topological space $(X, \tau)$, then $\bigcup_{\alpha} A_\alpha$ is a $\lambda_{\beta c}$-open set.

**Example 2.11.** Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \rightarrow P(X)$ as:
\[ \lambda(A) = \begin{cases} A & \text{if } A \neq \{a\}, \{b\} \\ X & \text{otherwise} \end{cases} \]

Now, we have \{a, b\} and \{b, c\} are \lambda_{\beta_c}-open sets, but \{a, b\} \cap \{b, c\} = \{b\} is not \lambda_{\beta_c}-open.

**Proposition 2.12.**[3] Let \( \lambda \) be an s-operation and s-regular. If \( A \) and \( B \) are \lambda_{\beta_c}-open sets in \( X \), then \( A \cap B \) is also a \lambda_{\beta_c}-open set.

**Proposition 2.13.**[3] Let \( X \) be a space and \( A \subseteq X \). Then \( A \) is a \lambda_{\beta_c}-closed subset of \( X \) if and only if \( \lambda_{\beta_c}D(A) \subseteq A \).

**Proposition 2.14.**[3] For subsets \( A, B \) of a space \( X \), the following statements are true.

1. \( A \subseteq \lambda_{\beta_c}Cl(A) \).
2. \( \lambda_{\beta_c}Cl(A) \) is a \lambda_{\beta_c}-closed set in \( X \).
3. \( \lambda_{\beta_c}Cl(A) \) is a smallest \lambda_{\beta_c}-closed set, which contain \( A \).
4. \( A \) is a \lambda_{\beta_c}-closed set if and only if \( A = \lambda_{\beta_c}Cl(A) \).
5. \( \lambda_{\beta_c}Cl(\phi) = \phi \) and \( \lambda_{\beta_c}Cl(X) = X \).
6. If \( A \) and \( B \) are subsets of the space \( X \) with \( A \subseteq B \). Then \( \lambda_{\beta_c}Cl(A) \subseteq \lambda_{\beta_c}Cl(B) \).
7. For any subsets \( A, B \) of a space \( X \). \( \lambda_{\beta_c}Cl(A) \cup \lambda_{\beta_c}Cl(B) \subseteq \lambda_{\beta_c}Cl(A \cup B) \).
8. For any subsets \( A, B \) of a space \( X \). \( \lambda_{\beta_c}Cl(A \cap B) \subseteq \lambda_{\beta_c}Cl(A) \cap \lambda_{\beta_c}Cl(B) \).

**Proposition 2.15**[3]. Let \( X \) be a space and \( A \subseteq X \). Then \( \lambda_{\beta_c}Cl(A) = A \cup \lambda_{\beta_c}D(A) \).

**Definition 2.16.**[4] A space \( X \) is said to be \( \lambda \)-connected if there does not exist a pair \( A, B \) of nonempty disjoint \( \lambda \)-open subset of \( X \) such that \( X = A \cup B \), otherwise \( X \) is called \( \lambda \)-disconnected. In this case, the pair \( (A, B) \) is called a \( \lambda \)-disconnection of \( X \).
Definition 2.17.[5]. A space $X$ is said to be $\lambda_c$-connected if there does not exist a pair $A$, $B$ of nonempty disjoint $\lambda_c$-open subset of $X$ such that $X = A \cup B$, otherwise $X$ is called $\lambda_c$-disconnected. In this case, the pair $(A, B)$ is called a $\lambda_c$-disconnection of $X$.

Theorem 2.17.[4]. Every $s$-connected space is $\lambda$-connected.

The converse of Theorem 2.17 is not true in general by the following example:

Example 2.18.[4]. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. We define an $s$-operation $\lambda: SO(X) \rightarrow P(X)$ as:

$$\lambda(A) = \begin{cases} X & \text{if } a \in A \\ A & \text{otherwise} \end{cases}$$

$SO(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}.$

$SO_\lambda(X) = \{\emptyset, \{b\}, X\}.$

We have $X$ is $\lambda$-connected, but it is not $s$-connected.

Theorem 2.19.[5]. Every $\lambda$-connected space is $\lambda_c$-connected.

Remark 2.20. We can show that, the converse of Theorem 2.19 is not true in general. In fact the space $X$ of Example 2.6 is $\lambda_c$-connected, but not $\lambda$-connected.

Corollary 2.21.[5]. Every $s$-connected space is $\lambda_c$-connected.

3. $\lambda_{fc}$-Connected Spaces

In this section, we define and study some characterizations and properties of a new space called $\lambda_{sc}$-connected space.

We start this section with the following definitions.

Definition 3.1. Let $X$ be a space and $Y \subseteq X$. Then the class of $\lambda_{fc}$-open sets in $Y$ ($SO_{\lambda_{fc}}(Y)$) is defined in a natural way as:
SO_{\lambda_{pc}}(Y) = \{ Y \cap V : V \in SO_{\lambda_{pc}}(X) \}.

That is \( W \) is \( \lambda_{pc} \)-open in \( Y \) if and only if \( W = Y \cap V \), where \( V \) is a \( \lambda_{pc} \)-open set in \( X \).

Thus, \( Y \) is a subspace of \( X \) with respect to \( \lambda_{pc} \)-open set.

**Definition 3.2.** A space \( X \) is said to be \( \lambda_{pc} \)-connected if there does not exist a pair \( A, B \) of nonempty disjoint \( \lambda_{pc} \)-open subset of \( X \) such that \( X = A \cup B \), otherwise \( X \) is called \( \lambda_{pc} \)-disconnected. In this case, the pair \( (A, B) \) is called a \( \lambda_{pc} \)-disconnection of \( X \).

**Definition 3.3.** Let \( X \) be a space and \( \lambda : SO(X) \rightarrow P(X) \) an s-operation, then the family \( SO_{\lambda_{pc}}(X) \) is called \( \lambda_{pc} \)-indiscrete space if \( SO_{\lambda_{pc}}(X) = \{ \phi, X \} \).

**Definition 3.4.** Let \( X \) be a space and \( \lambda : SO(X) \rightarrow P(X) \) an s-operation then the family \( SO_{\lambda_{pc}}(X) \) is called a \( \lambda_{pc} \)-discrete space if \( SO_{\lambda_{pc}}(X) = P(X) \).

**Example 3.5.** Every \( \lambda_{pc} \)-indiscrete space is \( \lambda_{c} \)-connected.

We give in below a characterization of \( \lambda_{pc} \)-connected spaces, the proof of which is straight forward.

**Theorem 3.6.** A space \( X \) is \( \lambda_{pc} \)-disconnected (resp. \( \lambda_{pc} \)-connected ) if and only if there exists (resp., does not exist) a non empty proper subset \( A \) of \( X \), which is both \( \lambda_{pc} \)-open and \( \lambda_{pc} \)-closed in \( X \).

**Theorem 3.7.** Every \( \lambda \)-connected space is \( \lambda_{pc} \)-connected.

**Proof.** Let \( X \) be \( \lambda \)-connected, then there does not exist a pair \( A, B \) of nonempty disjoint \( \lambda \)-open subset of \( X \) such that \( X = A \cup B \), but every \( \lambda_{pc} \)-open set is a \( \lambda \)-open set by Proposition 2.4, so there does not exist a pair \( A, B \) of nonempty disjoint \( \lambda_{pc} \)-open subset of \( X \) such that \( X = A \cup B \). Thus \( X \) is \( \lambda_{pc} \)-connected.
The converse of Theorem 3.7 is not true in general as it is shown by the following example:

**Example 3.8.** Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. We define an s-operation $\lambda : SO(X) \rightarrow P(X)$ as follows:

$$\lambda(A) = \begin{cases} A & \text{if } A = \{a\} \\ X & \text{otherwise} \end{cases}.$$  

$SO(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} = B_0(X)$.

$SO_\lambda(X) = \{\emptyset, \{a\}, X\}$.

$SO_{\lambda_{\beta c}}(X) = \{\emptyset, X\}$.

We have $X$ is $\lambda_{\beta c}$-connected, but it is not $\lambda$-connected.

**Theorem 3.9.** Every $\lambda_{\beta c}$-connected space is $\lambda_c$-connected.

**Proof.** Let $X$ be $\lambda_{\beta c}$-connected, then there does not exist a pair $A, B$ of nonempty disjoint $\lambda_{\beta c}$-open subset of $X$ such that $X = A \cup B$, but every $\lambda_c$-open set is a $\lambda_{\beta c}$-open set by Proposition 2.4, so there does not exist a pair $A, B$ of nonempty disjoint $\lambda_c$-open subset of $X$ such that $X = A \cup B$. Thus $X$ is $\lambda_c$-connected.

**Remark 3.10.** The converse of Theorem 3.9 is not true, in general. The space $X$ of Example 2.6 is $\lambda_c$-connected, but not $\lambda_{\beta c}$-connected.

**Remark 3.11.** The following diagram combining Theorem 2.17, Theorem 2.19, Theorem 3.7, Theorem 3.9, Corollary 2.21, Example 2.6, Example 2.18, Example 3.8, Remark 2.20 and Remark 3.10.
Definition 3.12. Let $X$ be a space and $A \subseteq X$. The $\lambda_{\beta_{c}}$-boundary of $A$, written $\lambda_{\beta_{c}} Bd(A)$, is defined as the set $\lambda_{\beta_{c}} Bd(A) = \lambda_{\beta_{c}} Cl(A) \cap \lambda_{\beta_{c}} Cl(X \setminus A)$.

Theorem 3.13. A space $X$ is $\lambda_{\beta_{c}}$-connected if and only if every nonempty proper subspace has a nonempty $\lambda_{\beta_{c}}$-boundary.

Proof. Suppose that a nonempty proper subspace $A$ of a $\lambda_{\beta_{c}}$-connected space $X$ has empty $\lambda_{\beta_{c}}$-boundary. Then $A$ is $\lambda_{\beta_{c}}$-open and $\lambda_{\beta_{c}} Cl(A) \cap \lambda_{\beta_{c}} Cl(X \setminus A) = \emptyset$. Let $p$ be a $\lambda_{\beta_{c}}$-limit point of $A$. Then $p \in \lambda_{\beta_{c}} Cl(A)$ but $p \notin \lambda_{\beta_{c}} Cl(X \setminus A)$. In particular $p \notin (X \setminus A)$ and so $p \in A$. Thus $A$ is $\lambda_{\beta_{c}}$-closed and $\lambda_{\beta_{c}}$-open. By Theorem 3.6, $X$ is $\lambda_{\beta_{c}}$-disconnected. This contradiction gives that $A$ has a nonempty $\lambda_{\beta_{c}}$-boundary.

Conversely, suppose $X$ is $\lambda_{\beta_{c}}$-disconnected. Then by Theorem 3.6, $X$ has a proper subspace $A$ which is both $\lambda_{\beta_{c}}$-closed and $\lambda_{\beta_{c}}$-open. Then $\lambda_{\beta_{c}} Cl(A) = A$, $\lambda_{\beta_{c}} Cl(X \setminus A) = (X \setminus A)$ and $\lambda_{\beta_{c}} Cl(A) \cap \lambda_{\beta_{c}} Cl(X \setminus A) = \emptyset$. So $A$ has empty $\lambda_{\beta_{c}}$-boundary, a contradiction. Hence $X$ is $\lambda_{\beta_{c}}$-connected. This completes the proof.

Theorem 3.14. Let $(A, B)$ be a $\lambda_{\beta_{c}}$-disconnection of a space $X$ and $C$ be a $\lambda_{\beta_{c}}$-connected subspace of $X$. Then $C$ is contained in $A$ or in $B$.

Proof. Suppose that $C$ is neither contained in $A$ nor in $B$. Then $C \cap A$, $C \cap B$ are both nonempty $\lambda_{\beta_{c}}$-open subsets of $C$ such that $(C \cap A) \cap (C \cap B) = \emptyset$ and $(C \cap A) \cup (C \cap B) = C$. This gives that $(C \cap A, C \cap B)$ is a $\lambda_{\beta_{c}}$-disconnection of $C$. This contradiction proves the theorem.

Theorem 3.15. Let $X = \bigcup_{\alpha \in I} X_{\alpha}$, where each $X_{\alpha}$ is $\lambda_{\beta_{c}}$-connected and $\bigcap_{\alpha \in I} X_{\alpha} \neq \emptyset$. Then $X$ is $\lambda_{\beta_{c}}$-connected.

Proof. Suppose on the contrary that $(A, B)$ is a $\lambda_{\beta_{c}}$-disconnection of $X$. Since each $X_{\alpha}$ is $\lambda_{\beta_{c}}$-connected, therefore by Theorem 3.14, $X_{\alpha} \subseteq A$ or $X_{\alpha} \subseteq B$. Since
\[ \bigcap_{\alpha \in I} X_{\alpha} \neq \emptyset, \text{ therefore all } X_{\alpha} \text{ are contained in } A \text{ or in } B. \] This gives that, if \( X \subseteq A \), then \( B = \emptyset \) or if \( X \subseteq B \), then \( A = \emptyset \). This contradiction proves that \( X \) is \( \lambda_{\beta_c} \)-connected. Which completes the proof.

Using Theorem 3.15, we give a characterization of \( \lambda_{\beta_c} \)-connectedness as follows:

**Theorem 3.16.** A space \( X \) is \( \lambda_{\beta_c} \)-connected if and only if for every pair of points \( x, y \) in \( X \), there is a \( \lambda_{\beta_c} \)-connected subset of \( X \), which contains both \( x \) and \( y \).

**Proof.** The necessity is immediate since the \( \lambda_{\beta_c} \)-connected space itself contains these two points.

For the sufficiency, suppose that for any two points \( x, y \); there is a \( \lambda_{\beta_c} \)-connected subspace \( C_{x,y} \) of \( X \) such that \( x, y \in C_{x,y} \). Let \( a \in X \) be a fixed point and \( \{C_{a,x}, x \in X\} \) a class of all \( \lambda_{\beta_c} \)-connected subsets of \( X \), which contain \( a \) and \( x \in X \).

Then \( X = \bigcup_{x \in X} C_{a,x} \) and \( \bigcap_{x \in X} C_{a,x} \neq \emptyset \). Therefore, by Theorem 3.15, \( X \) is \( \lambda_{\beta_c} \)-connected. This completes the proof.

**Theorem 3.17.** Let \( C \) be a \( \lambda_{\beta_c} \)-connected subset of a space \( X \) and \( A \subseteq X \) such that \( C \subseteq A \subseteq \lambda_{\beta_c} Cl(C) \). Then \( A \) is \( \lambda_{\beta_c} \)-connected.

**Proof.** It is sufficient to show that \( \lambda_{\beta_c} Cl(C) \) is \( \lambda_{\beta_c} \)-connected. On the contrary, suppose that \( \lambda_{\beta_c} Cl(C) \) is \( \lambda_{\beta_c} \)-disconnected. Then there exists a \( \lambda_{\beta_c} \)-disconnection \((H, K)\) of \( \lambda_{\beta_c} Cl(C) \). That is, \( H \cap C, K \cap C \) are \( \lambda_{\beta_c} \)-open sets in \( C \) such that \((H \cap C) \cap (K \cap C) = (H \cap K) \cap C = \emptyset \) and \((H \cap C) \cap (K \cap C) = (H \cap K) \cap C = C \). This gives that \((H \cap C, K \cap C)\) is a \( \lambda_{\beta_c} \)-disconnection of \( C \), a contradiction. This proves that \( \lambda_{\beta_c} Cl(C) \) is \( \lambda_{\beta_c} \)-connected.
4. \( \lambda_{\beta_c} \)-component of a set

We introduce the following definition

**Definition 4.1.** A maximal \( \lambda_{\beta_c} \)-connected subset of a space \( X \) is called a \( \lambda_{\beta_c} \)-component of \( X \). If \( X \) itself is \( \lambda_{\beta_c} \)-connected, then \( X \) is the only \( \lambda_{\beta_c} \)-component of \( X \).

Next we study the properties of \( \lambda_{\beta_c} \)-components of a space \( X \):

**Theorem 4.2.** Let \( (X, \tau) \) be a topological space. Then

1. For each \( x \in X \), there is exactly one \( \lambda_{\beta_c} \)-component of \( X \) containing \( x \).
2. Each \( \lambda_{\beta_c} \)-connected subset of \( X \) is contained in exactly one \( \lambda_{\beta_c} \)-component of \( X \).
3. A \( \lambda_{\beta_c} \)-connected subset of \( X \), which is both \( \lambda_{\beta_c} \)-open and \( \lambda_{\beta_c} \)-closed is a \( \lambda_{\beta_c} \)-component, if \( \lambda \) is s-regular.
4. Every \( \lambda_{\beta_c} \)-component of \( X \) is \( \lambda_{\beta_c} \)-closed in \( X \).

**Proof:**

1. Let \( x \in X \) and \( \{C_\alpha : \alpha \in I\} \) be a class of all \( \lambda_{\beta_c} \)-connected subsets of \( X \) containing \( x \). Put \( C = \bigcup_{\alpha \in I} C_\alpha \), then by Theorem 3.15, \( C \) is \( \lambda_{\beta_c} \)-connected and \( x \in X \). Suppose \( C \subseteq C^* \), for some \( \lambda_{\beta_c} \)-connected subset \( C^* \) of \( X \). Then \( x \in C^* \) and hence \( C^* \) is one of the \( C_\alpha \)'s and hence \( C^* \subseteq C \). Consequently \( C = C^* \). This proves that \( C \) is a \( \lambda_{\beta_c} \)-component of \( X \), which contains \( x \).

2. Let \( A \) be a \( \lambda_{\beta_c} \)-connected subset of \( X \), which is not a \( \lambda_{\beta_c} \)-component of \( X \). Suppose that \( C_1, C_2 \) are \( \lambda_{\beta_c} \)-components of \( X \) such that \( A \subseteq C_1, A \subseteq C_2 \).

Since \( C_1 \cap C_2 = \emptyset \), \( C_1 \cup C_2 \) is another \( \lambda_{\beta_c} \)-connected set which contains \( C_1 \) as well as \( C_2 \), this contradicts the fact that \( C_1 \) and \( C_2 \) are \( \lambda_{\beta_c} \)-components. This proves that \( A \) is contained in exactly one \( \lambda_{\beta_c} \)-component of \( X \).
(3) Suppose that $A$ is a $\lambda_{\beta_c}$-connected subset of $X$ which is both $\lambda_{\beta_c}$-open and $\lambda_{\beta_c}$-closed. By (2), $A$ is contained in exactly one $\lambda_{\beta_c}$-component $C$ of $X$. If $A$ is a proper subset of $C$, and since $\lambda$ is s-regular, therefore $C = (C \cap A) \cup (C \cap (X \setminus A))$ is a $\lambda_{\beta_c}$-disconnection of $C$, a contradiction. Thus, $A = C$.

(4) Suppose a $\lambda_{\beta_c}$-component $C$ of $X$ is not $\lambda_{\beta_c}$-closed. Then, by Theorem 3.17, $\lambda_{\beta_c}Cl(A)$ is $\lambda_{\beta_c}$-connected containing a $\lambda_{\beta_c}$-component $C$ of $X$. This implies $C = \lambda_{\beta_c}Cl(A)$ and hence $C$ is $\lambda_{\beta_c}$-closed. This completes the proof.$\blacksquare$

We introduce the following definition

**Definition 4.3.** A space $X$ is said to be locally $\lambda_{\beta_c}$-connected if for any point $x \in X$ and any $\lambda_{\beta_c}$-open set $U$ containing $x$, there is a $\lambda_{\beta_c}$-connected $\lambda_{\beta_c}$-open set $V$ such that $x \in V \subseteq U$.

**Theorem 4.4.** A $\lambda_{\beta_c}$-open subset of $\lambda_{\beta_c}$-locally connected space is $\lambda_{\beta_c}$-locally connected.

**Proof.** Let $U$ be a $\lambda_{\beta_c}$-open subset of a $\lambda_{\beta_c}$-locally connected space $X$. Let $x \in U$ and $V$ a $\lambda_{\beta_c}$-open nbd of $x$ in $U$. Then $V$ is a $\lambda_{\beta_c}$-open nbd of $x$ in $X$. Since $X$ is $\lambda_{\beta_c}$-locally connected, therefore there exists a $\lambda_{\beta_c}$-connected, $\lambda_{\beta_c}$-open nbd $W$ of $x$ such that $x \in W \subseteq V$. So that $W$ is also a $\lambda_{\beta_c}$-connected $\lambda_{\beta_c}$-open nbd $x$ in $U$ such that $x \in W \subseteq U \subseteq V$ or $x \in W \subseteq V$. This proves that $U$ is $\lambda_{\beta_c}$-locally connected.$\blacksquare$
References